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Common structure of several completely integrable non-linear equations

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Abstract. We consider several non-linear equations which are known to be completely integrable by the method of inverse scattering transform, ranging from sine-Gordon and the Dodd-Bullough equation to the axisymmetric, stationary, Einstein-Maxwell equations.

Such equations are known to be invariant under a fundamental Lie group of symmetry (G), such as SL(2) or SL(3) in the simpler cases. Our treatment starts with the reformulation of the given equations as a system involving the scalar invariants of the Lie group (G) only; this scalar formulation is not only (G) invariant, but also manifestly invariant under the continuous group associated with the emergence of a spectral parameter, and is thus expected to be more fundamental and simpler. In all the cases that we have considered, including the Einstein-Maxwell system, vector pseudopotentials of dimension *not higher than three* have been derived; Bäcklund transformations have been found, which generate multisolitons starting from the vacuum, and have the general (reciprocal) form:

$$X' = 1/X$$

where X is a component of one of the vector (or tensor) pseudopotentials.

We also mention the result that Kinnersley's formulation of the Ernst (vacuum) equation is formally equivalent to the fluid dynamical problem of unidimensional ideal gas flow; the Dodd-Bullough equation is equivalent to another particular case of gas motion.

1. Introduction

The basic properties of the evolution equations which are integrable by the method of inverse scattering transform (IST) have been analysed by many authors and are well known (for reviews, see, e.g., [1-5]). Such equations are considered to be 'completely integrable' and are also characterised by an infinite-dimensional Lie symmetry group, which usually arises from the combination of a finite-dimensional Lie group (G), such as SL(N), and a discrete transformation: the Bäcklund transformation.

We show in the present paper that the analysis of many integrable systems in two independent variables can be done in a clarifying and rather unified way by first forming the invariants of the Lie group (G) and then selecting them as the basic unknown functions, Ω.

The corresponding pseudopotential is, in most cases, a three-component vector $\boldsymbol{X}$, given by a set of three linear second-order PDE, of the general form:

$$\begin{pmatrix} \boldsymbol{X}_{\alpha\alpha} \\ \boldsymbol{X}_{\alpha\beta} \\ \boldsymbol{X}_{\beta\beta} \end{pmatrix} = M \begin{pmatrix} \boldsymbol{X}_{\alpha} \\ \boldsymbol{X}_{\beta} \\ \boldsymbol{X} \end{pmatrix} \tag{1.1}$$

where the 3×3 matrix M has coefficients which are a function of Ω; the subscripts denote differentiation with respect to the two independent variables α, β. The Lie group (G) may be the full SL(3) group of the (unimodular) linear transformations of $\boldsymbol{X}$, or—as in the sine-Gordon case—a subgroup of it. This formulation (1.1) is susceptible of generalisation (§ 2.1) to accommodate Lie groups (G) of order higher than SL(3).

By a process of elimination, overdetermined pairs of *ordinary differential equations* (ODE) can be derived and provide an effective way of computing the pseudopotentials associated with any given solution Ω, since the equations are linear. As the equations are uncoupled, they can be written down for each component X of $\boldsymbol{X}$ separately.

The Bäcklund transformation (BT) is yet another symmetry of the system, one that does *not* leave Ω invariant; its transformation formulae can be derived in closed form, in terms of Ω and its pseudopotentials $\boldsymbol{X}$:

$$\Omega' = \Omega'(\Omega; \boldsymbol{X})$$
$$X'_i = X'_i(\Omega; \boldsymbol{X}).$$

In all cases that have been considered here, *the BT turns out to be a reciprocal transformation*: applying it twice gives back the original solution $(\Omega; X)$, *provided that the pseudopotential X' is not independently recalculated* before applying the transformation a second time. Thus, e.g., the BT in the axisymmetric Einstein-Maxwell case is just the reciprocal transformation which exchanges the time and azimuthal coordinates t, φ (§ 6); yet, in combination with the Lie group (G), the BT does generate the infinite Lie group of symmetry characteristic of completely integrable systems. It also generates the n-soliton formula, starting from the vacuum.

The formulation (1.1) has already been shown to be appropriate to the analysis of the Dodd-Bullough equation [6]; the corresponding Lie group (G) is SL(3) and the Bäcklund transformation is simply

$$X' = 1/X.$$

In § 2 we present a few general results concerning the properties of systems of the form (1.1) and briefly recall the essentials of the analysis of the Dodd-Bullough (DB) case. In § 3 the sine-Gordon equation is seen to be amenable to *a very closely similar treatment* and the classical Bäcklund transformation for sine-Gordon is shown to be given by *the same formula*: $X' = 1/X$. An interesting peculiarity is that the formulation (1.1) still retains the SL(3) symmetry, as in the DB case, but the applicability of the BT requires that an additional non-linear constraint on X be satisfied, and that is compatible with an SL(2) subgroup only.

In § 4 we consider the axisymmetric vacuum Einstein equations, which are represented by the Ernst equations [7]; their analysis closely parallels that of sine-Gordon, which indeed constitutes the limiting form of the Ernst equations far away from the symmetry axis. In § 5 we introduce a generalised sine-Gordon equation with genuine SL(3) symmetry, which may be called sine-Gordon-Maxwell, as it similarly constitutes the limiting form of the Einstein-Maxwell equations away from the axis and, in § 6, we generalise our treatment to include the axisymmetric Einstein-Maxwell system.

In *all* the above-mentioned cases, a *three-dimensional* vector pseudopotential $\boldsymbol{X}$ turns out to be sufficient and can be effectively calculated by solving linear ODE which are of the third order only, instead of the cumbersome eighth order that might have been expected from the dimension of the usual Einstein-Maxwell linear representation [8] (§§ 5 and 6). This simplification results from the *separability* of the equations defining the pseudopotentials; the usual eight-dimensional representation separates into a pair of (complex conjugate) fundamental representations of dimension 3.

2. General formalism, with application to the Dodd–Bullough equation

2.1. *General formulation, with N-dimensional pseudopotentials*

The general principle underlying the present method may be stated as follows: first, we look for a base $(\boldsymbol{E}_1, \ldots, \boldsymbol{E}_n)$ of the simplest available representation of the Lie group (G) that can be obtained explicitly in terms of the invariant unknown function Ω and of the associated (pseudo)potentials; in the Einstein–Maxwell case, for instance (see § 6), that is the eight-dimensional representation in terms of the four non-linear Ernst potentials proposed by Kinnersley [8]. Then we expand the first-order derivatives of $\boldsymbol{E}_i$ along that base, in the form:

$$\begin{pmatrix} \boldsymbol{E}_{1\alpha} \\ \vdots \\ \boldsymbol{E}_{N\alpha} \end{pmatrix} = M \begin{pmatrix} \boldsymbol{E}_1 \\ \vdots \\ \boldsymbol{E}_N \end{pmatrix} \tag{2.1a}$$

$$\begin{pmatrix} \boldsymbol{E}_{1\beta} \\ \vdots \\ \boldsymbol{E}_{N\beta} \end{pmatrix} = \tilde{M} \begin{pmatrix} \boldsymbol{E}_1 \\ \vdots \\ \boldsymbol{E}_N \end{pmatrix} \tag{2.1b}$$

where M, $\tilde{M}$ are two $N \times N$ matrices whose coefficients are all (G) invariant and are expressible in terms of the invariant unknown function(s) Ω; α, β are the two independent variables. Each equation, (2.1a) and (2.1b), is equivalent to an Nth-order ordinary differential equation for any of the base vectors, $\boldsymbol{E}_i$ say, since all $(N-1)$ other vectors may easily be eliminated and, since the order is greater than unity, each $\boldsymbol{E}_i$ plays the role of a *pseudopotential*, according to the definition given by Wahlquist and Estabrook [9].

The system (2.1) being an overdetermined system, there are conditions for the integrability of $(\boldsymbol{E})$, which arise from

$$\partial_\beta(\boldsymbol{E}_\alpha) = M_\beta(\boldsymbol{E}) + M\tilde{M}(\boldsymbol{E})$$
$$\partial_\alpha(\boldsymbol{E}_\beta) = \tilde{M}_\alpha(\boldsymbol{E}) + \tilde{M}M(\boldsymbol{E})$$

where $(\boldsymbol{E})$ denotes the column: $(\boldsymbol{E}_1, \ldots, \boldsymbol{E}_N)$. Thus the conditions are

$$\tilde{M}_\alpha - M_\beta = [M, \tilde{M}]. \tag{2.2}$$

2.2. *Three-dimensional pseudopotentials*

It is one of the results of the present study that *three-dimensional* representations are in fact explicitly obtainable even in such comparatively complex cases as the axisymmetric Einstein–Maxwell equations, whose symmetry group (G) is of order no higher than SL(3). In such a case we rewrite $\boldsymbol{E}_1 \equiv \boldsymbol{X}$ and choose $\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{X}_\beta$ for the base vectors [10], so that the formulation (2.1) reduces to a set of three second-order PDE for the pesudopotential $\boldsymbol{X}$:

$$\boldsymbol{X}_{\alpha\alpha} = b_0\boldsymbol{X}_\alpha + B\boldsymbol{X}_\beta + b_1\boldsymbol{X} \tag{2.3a}$$

$$\boldsymbol{X}_{\beta\beta} = C\boldsymbol{X}_\alpha + c_0\boldsymbol{X}_\beta + c_1\boldsymbol{X} \tag{2.3b}$$

$$\boldsymbol{X}_{\alpha\beta} = a_0\boldsymbol{X}_\alpha + a_1\boldsymbol{X}_\beta + D\boldsymbol{X}. \tag{2.3c}$$

The integrability conditions are derived from $\partial_\beta(\boldsymbol{X}_{\alpha\alpha})=\partial_\alpha(\boldsymbol{X}_{\alpha\beta}), \partial_\alpha(\boldsymbol{X}_{\beta\beta})=\partial_\beta(\boldsymbol{X}_{\alpha\beta})$, and are the following:

$$a_{0\alpha}-b_{0\beta}=BC-D-a_0a_1 \tag{2.4a}$$

$$a_{1\alpha}-B_\beta=B(c_0-a_0)+a_1(b_0-a_1)+b_1 \tag{2.4b}$$

$$D_\alpha-b_{1\beta}=Bc_1+D(b_0-a_1)-a_0b_1 \tag{2.4c}$$

$$a_{1\beta}-c_{0\alpha}=BC-D-a_0a_1 \tag{2.5a}$$

$$a_{0\beta}-C_\alpha=C(b_0-a_1)+a_0(c_0-a_0)+c_1 \tag{2.5b}$$

$$D_\beta-c_{1\alpha}=Cb_1+D(c_0-a_0)-a_1c_1. \tag{2.5c}$$

We remark that the above equations (2.4a) and (2.5a) ensure existence of a quantity ω, defined by

$$\begin{aligned}\omega_\alpha&=a_1+b_0\\ \omega_\beta&=a_0+c_0.\end{aligned} \tag{2.6}$$

In agreement with the general conclusions of the preceding section, $\boldsymbol{X}$ satisfies a pair of *third-order* ODE:

$$\boldsymbol{X}_{\alpha\alpha\alpha}=b_0\boldsymbol{X}_{\alpha\alpha}+(Ba_0+b_{0\alpha}+b_1)\boldsymbol{X}_\alpha+(B_\alpha+Ba_1)\boldsymbol{X}_\beta+(BD+b_{1\alpha})\boldsymbol{X} \tag{2.7}$$

where the expression (2.3a) in terms of α derivatives of $\boldsymbol{X}$ should be substituted for $\boldsymbol{X}_\beta$.

2.3. Reformulation as a pair of first-order equations for two dual vectors X, U

First, introducing the notation Ω for the triple product of the three-dimensional vectors $\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{X}_\beta$:

$$\Omega\equiv(\boldsymbol{X},\boldsymbol{X}_\alpha,\boldsymbol{X}_\beta)\equiv\boldsymbol{X}\cdot(\boldsymbol{X}_\alpha\wedge\boldsymbol{X}_\beta)$$

we note the following identification of the matrix coefficients in terms of triple products:

$$\begin{aligned}
&B\Omega=(\boldsymbol{X},\boldsymbol{X}_\alpha,\boldsymbol{X}_{\alpha\alpha}) && b_1\Omega=(\boldsymbol{X}_\alpha,\boldsymbol{X}_\beta,\boldsymbol{X}_{\alpha\alpha}) && b_0\Omega=-(\boldsymbol{X},\boldsymbol{X}_\beta,\boldsymbol{X}_{\alpha\alpha})\\
&C\Omega=-(\boldsymbol{X},\boldsymbol{X}_\beta,\boldsymbol{X}_{\beta\beta}) && c_1\Omega=(\boldsymbol{X}_\alpha,\boldsymbol{X}_\beta,\boldsymbol{X}_{\beta\beta}) && c_0\Omega=(\boldsymbol{X},\boldsymbol{X}_\alpha,\boldsymbol{X}_{\beta\beta})\\
&D\Omega=(\boldsymbol{X}_\alpha,\boldsymbol{X}_\beta,\boldsymbol{X}_{\alpha\beta})\\
&a_0=\partial_\beta(\ln\Omega)-c_0 && a_1=\partial_\alpha(\ln\Omega)-b_0.
\end{aligned} \tag{2.8}$$

Thus, the quantity ω defined above (see equation (2.6)) is just

$$\omega\equiv\ln\Omega.$$

Let us now introduce a new vector $\boldsymbol{U}$, the three-dimensional (Euclidean) cross-product:

$$\boldsymbol{U}\equiv\frac{1}{\Omega}\boldsymbol{X}_\alpha\wedge\boldsymbol{X}_\beta. \tag{2.9}$$

It is easily seen that $\boldsymbol{U}$ satisfies the following system:

$$\begin{aligned}
\Omega\boldsymbol{U}_\alpha&=-D\boldsymbol{X}\wedge\boldsymbol{X}_\alpha+b_1\boldsymbol{X}\wedge\boldsymbol{X}_\beta\\
\Omega\boldsymbol{U}_\beta&=-c_1\boldsymbol{X}\wedge\boldsymbol{X}_\alpha+D\boldsymbol{X}\wedge\boldsymbol{X}_\beta\\
\boldsymbol{U}\cdot\boldsymbol{X}&=1.
\end{aligned} \tag{2.10}$$

This may be solved as a linear system for $\boldsymbol{X}_\alpha$ and $\boldsymbol{X}_\beta$ and yields a dual system of the same form:

$$\begin{aligned}\tilde{\Omega}\boldsymbol{X}_\alpha &= -D\boldsymbol{U}\wedge\boldsymbol{U}_\alpha + b_1\boldsymbol{U}\wedge\boldsymbol{U}_\beta\\ \tilde{\Omega}\boldsymbol{X}_\beta &= -c_1\boldsymbol{U}\wedge\boldsymbol{U}_\alpha + D\boldsymbol{U}\wedge\boldsymbol{U}_\beta\end{aligned}\tag{2.11}$$

where, by definition of $\tilde{\Omega}$,

$$\Omega\tilde{\Omega} = (b_1c_1 - D^2).\tag{2.12}$$

2.4. An application: the Dodd-Bullough equation

That is the case where equations (2.3) for $\boldsymbol{X}$ assume the form:

$$\begin{aligned}\boldsymbol{X}_{\alpha\alpha} &= \omega_\alpha\boldsymbol{X}_\alpha + B\boldsymbol{X}_\beta\\ \boldsymbol{X}_{\beta\beta} &= \omega_\beta\boldsymbol{X}_\beta + C\boldsymbol{X}_\alpha\\ \boldsymbol{X}_{\alpha\beta} &= \Omega\boldsymbol{X}\end{aligned}\tag{2.13}$$

where we have used the lower-case symbol ω for the logarithm, $\ln\Omega$. Then the integrability conditions (2.4) and (2.5) are

$$\begin{aligned}&\omega_{\alpha\beta} + BC - \Omega = 0\\ &B_\beta + B\omega_\beta = 0\end{aligned}$$

and

$$\begin{aligned}&\omega_{\alpha\beta} + BC - \Omega = 0\\ &C_\alpha + C\omega_\alpha = 0.\end{aligned}$$

The equations for B and C are obviously integrable in the form: $B = F(\alpha)/\Omega$, $C = G(\beta)/\Omega$, and one obtains a second-order PDE for the single unknown Ω (we will, for clarity, retain the symbol ∂ for denoting differentiation of relatively complex expressions; i.e. we will write $\partial_{\alpha\beta}(\ldots)$ rather than $(\ldots)_{\alpha\beta}$ when $(\ldots)$ involves more than just one symbol):

$$\partial_{\alpha\beta}(\ln\Omega) = \Omega - F(\alpha)G(\beta)/\Omega^2.$$

As observed in [6], this is obviously reducible, by a conformal transformation, to the form

$$\partial_{\alpha\beta}(\ln\Omega) = \Omega - 1/\Omega^2\tag{2.14}$$

which, as announced, is the Dodd-Bullough equation [11].

The associated first-order system for the two dual vectors $\boldsymbol{U}$, $\boldsymbol{X}$ has a particularly elegant and symmetrical form:

$$\begin{aligned}\boldsymbol{U}_\alpha &= \Gamma\boldsymbol{X}\wedge\boldsymbol{X}_\alpha\\ \boldsymbol{U}_\beta &= -\Gamma\boldsymbol{X}\wedge\boldsymbol{X}_\beta\\ \boldsymbol{U}\cdot\boldsymbol{X} &= 1\end{aligned}\tag{2.15}$$

or, equivalently,

$$\begin{aligned}\boldsymbol{X}_\alpha &= -\Gamma^{-1}\boldsymbol{U}\wedge\boldsymbol{U}_\alpha\\ \boldsymbol{X}_\beta &= \Gamma^{-1}\boldsymbol{U}\wedge\boldsymbol{U}_\beta\\ \boldsymbol{U}\cdot\boldsymbol{X} &= 1\end{aligned}\tag{2.16}$$

where Γ is a constant. It was first noted in [12] that Euler equations of one-dimensional fluid flow with arbitrary equation of state are of the general form (2.15) and (2.16), with a factor Γ determined by the equation of state and generally non-constant. Complete integrability obtains for the particular equation of state associated with a constant Γ (see [13]).

That is not the sole case of complete integrability, however. As we shall see, the sine-Gordon equation is also of this type, with a Γ equal to

$$\Gamma_{SG} = 1/\boldsymbol{X}^2.$$

The Ernst equations, of which the Kerr metric associated with a rotating black hole is a particular solution [7], are of the same type too, as shown in § 4, with

$$\Gamma_{\text{Ernst}} = \rho/\boldsymbol{X}^2 \qquad (\rho = \alpha - \beta).$$

2.5. Analysis of the Dodd-Bullough equation: the pseudopotential and Bäcklund transformation

The Dodd-Bullough equation was originally shown by Mikhailov [14] to be integrable by the inverse scattering transform. Gaffet [6] obtained the Bäcklund transformation (which had been thought to be absent in the DB case; see [5]). We briefly summarise here the main points of this analysis.

The DB equation (2.14) is easily seen to be the integrability condition for the existence of a pseudopotential X, defined as

$$\Omega X_{\alpha\alpha} = \Omega_\alpha X + \lambda^3 X_\beta \tag{2.17a}$$

$$\Omega X_{\beta\beta} = \Omega_\beta X + X_\alpha/\lambda^3 \tag{2.17b}$$

$$X_{\alpha\beta} = \Omega X \tag{2.17c}$$

where the spectral parameter λ has been introduced so as to restore conformal invariance. We may eliminate Ω from the last equation and there results a pair of third-order cubic homogeneous PDE for X (see [6], equation (4.9)):

$$\begin{aligned} &X_\alpha X_{\alpha\alpha\beta} - X_{\alpha\beta}(X_{\alpha\alpha} + X_\alpha^2/X) + \lambda^3 X X_\beta = 0 \\ &X_\beta X_{\alpha\beta\beta} - X_{\alpha\beta}(X_{\beta\beta} + X_\beta^2/X) + X X_\alpha/\lambda^3 = 0. \end{aligned} \tag{2.18}$$

As the equations are homogeneous, it is natural to rewrite them in terms of $x \equiv \ln X$; the resulting equations turn out to have a definite parity in x and λ and are thus invariant under the transformation:

$$X' = 1/X. \tag{2.19}$$

The corresponding transformation formula for Ω is, using $\Omega \equiv X_{\alpha\beta}/X$,

$$\Omega + \Omega' = 2X_\alpha X_\beta / X^2 \tag{2.20}$$

which, taking account of (2.17*c*), is in the form of a truncated Painlevé expansion. Equations (2.19) and (2.20) constitute the Bäcklund transformation for the DB equation; it does include a spectral parameter, which enters equations (2.17) defining the pseudopotential X. The transformation looks reciprocal at first sight, but it is not if the pseudopotential X is recalculated at each step, using (2.17); in other words, the combination of the BT with the SL(3) group of linear transformations of X is an infinite group.

The above BT was shown in [6] to yield the soliton solution when applied to the vacuum, and the two-interacting soliton formula when applied to an isolated soliton, in agreement with the usual pattern of multisoliton construction from BT.

It is worth noting that, since the DB equation allows a fluid dynamical interpretation as pointed out in the preceding section, the BT must have a hydrodynamical analogue: i.e. the symmetry discovered around 1954 by Martin and Ludford [15] and Stanyukovitch [16], which transforms the integrated mass $M=\int \rho\, dx$ into its inverse: $M'=1/M$, without affecting the time coordinate. An interpretation in terms of a space-dependent rescaling of the fundamental units was proposed in [17].

3. The sine-Gordon equation

The occurrence of 3D vector pseudopotentials in the sine-Gordon case is related to the fact that sine-Gordon is equivalent to the bidimensional O(3) non-linear σ model [10, 18].

3.1. A derivation

We start with the general form (2.15), choosing a factor Γ of the form:

$$\begin{aligned}\Gamma &= -1/\boldsymbol{X}^2\\ &= -1/(X_1^2+X_2^2+X_3^2).\end{aligned} \tag{3.1}$$

It turns out to be convenient to rewrite the equations in terms of the unit vector $\boldsymbol{X}' \equiv \boldsymbol{X}/\sqrt{\boldsymbol{X}^2}$:

$$\begin{aligned}\boldsymbol{U}_\alpha &= -\boldsymbol{X}' \wedge \boldsymbol{X}'_\alpha\\ \boldsymbol{U}_\beta &= \boldsymbol{X}' \wedge \boldsymbol{X}'_\beta\\ \boldsymbol{X}'^2 &= 1.\end{aligned} \tag{3.2}$$

We forsake the prime and rewrite $\boldsymbol{X}$ and $\boldsymbol{X}'$ from now on; solving for $\boldsymbol{X}_\alpha$, $\boldsymbol{X}_\beta$ in the above equations produces the dual system:

$$\boldsymbol{X}_\alpha = \boldsymbol{X} \wedge \boldsymbol{U}_\alpha \tag{3.3a}$$

$$\begin{aligned}\boldsymbol{X}_\beta &= -\boldsymbol{X} \wedge \boldsymbol{U}_\beta\\ \boldsymbol{X}^2 &= 1.\end{aligned} \tag{3.3b}$$

All four vectors $\boldsymbol{X}_\alpha$, $\boldsymbol{X}_\beta$, $\boldsymbol{U}_\alpha$, $\boldsymbol{U}_\beta$ thus belong to the two-dimensional plane orthogonal to $\boldsymbol{X}$; furthermore, $\{\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{U}_\alpha\}$ and $\{\boldsymbol{X}, \boldsymbol{X}_\beta, \boldsymbol{U}_\beta\}$ constitute two orthogonal trihedrals, rotated from one another by a certain angle φ. That angle will be identified with the sine-Gordon field when the equation is written in its standard form, $\varphi_{\alpha\beta} = \text{constant} \times \sin\varphi$.

The following geometrical relations hold:

$$\begin{aligned}&\boldsymbol{U}_\alpha \cdot \boldsymbol{X}_\beta = \boldsymbol{U}_\beta \cdot \boldsymbol{X}_\alpha = -(\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{X}_\beta) \equiv -\Omega\\ &\boldsymbol{X}_\alpha \wedge \boldsymbol{X}_\beta = -\boldsymbol{U}_\alpha \wedge \boldsymbol{U}_\beta = \Omega \boldsymbol{X}\\ &\boldsymbol{U}_\alpha^2 = \boldsymbol{X}_\alpha^2 \qquad \boldsymbol{U}_\beta^2 = \boldsymbol{X}_\beta^2\\ &\boldsymbol{X}_\alpha \wedge \boldsymbol{U}_\beta = -\boldsymbol{X}_\beta \wedge \boldsymbol{U}_\alpha = H\boldsymbol{X}\end{aligned}$$

and define two scalars, Ω and H.

The integrability conditions associated with (3.2) and (3.3) are respectively

$$\begin{aligned} \boldsymbol{X}_{\alpha\beta} &= -H\boldsymbol{X} \\ \boldsymbol{U}_{\alpha\beta} &= -\boldsymbol{U}_\alpha \wedge \boldsymbol{U}_\beta = \Omega \boldsymbol{X}. \end{aligned} \tag{3.4}$$

As a result we obtain the pair of first integrals:

$$\boldsymbol{X}_\alpha^2 = F(\alpha)$$
$$\boldsymbol{X}_\beta^2 = G(\beta).$$

Since the original system is conformally invariant, we may, without loss of generality [10], choose $F(\alpha) = \lambda^2$, $G(\beta) = 1/\lambda^2$, i.e.

$$\begin{aligned} \boldsymbol{X}_\alpha^2 &= \lambda^2 = \boldsymbol{U}_\alpha^2 \\ \boldsymbol{X}_\beta^2 &= 1/\lambda^2 = \boldsymbol{U}_\beta^2 \end{aligned} \tag{3.5}$$

where λ plays the role of a spectral parameter. Finally one obtains a system of second-order PDE for $\boldsymbol{X}$, of the general form (2.3):

$$\begin{aligned} \boldsymbol{X}_{\alpha\alpha} &= \omega_\alpha(\boldsymbol{X}_\alpha - \lambda^2 \boldsymbol{X}_\beta / H) - \lambda^2 \boldsymbol{X} \\ \boldsymbol{X}_{\beta\beta} &= \omega_\beta(\boldsymbol{X}_\beta - \boldsymbol{X}_\alpha / \lambda^2 H) - \boldsymbol{X}/\lambda^2 \\ \boldsymbol{X}_{\alpha\beta} &= -H\boldsymbol{X} \end{aligned} \tag{3.6}$$

where $\omega \equiv \ln \Omega$, H and Ω are constrained by $H^2 + \Omega^2 = 1$ and can be identified with

$$H = \cos\varphi \qquad \Omega = \sin\varphi. \tag{3.7}$$

Applying the general results of § 2 (see (2.4) and (2.5)), we find the equation satisfied by H and Ω:

$$\omega_{\alpha\beta} + \omega_\alpha \omega_\beta / H^2 + H = 0 \tag{3.8}$$

or, in terms of φ only,

$$\varphi_{\alpha\beta} = -\sin\varphi \tag{3.9}$$

which is the sine-Gordon equation, as desired.

3.2. *The Bäcklund transformation*

We may rewrite the system (3.6) in terms of φ:

$$\sin\varphi(X_{\alpha\alpha} + \lambda^2 X) = \varphi_\alpha(\cos\varphi X_\alpha - \lambda^2 X_\beta) \tag{3.10a}$$

$$\sin\varphi(X_{\beta\beta} + X/\lambda^2) = \varphi_\beta(\cos\varphi X_\beta - X_\alpha/\lambda^2) \tag{3.10b}$$

$$X_{\alpha\beta} + \cos\varphi X = 0. \tag{3.10c}$$

In the same way as in the DB case, $\cos\varphi \equiv -X_{\alpha\beta}/X$ may be eliminated and the result of the elimination is a pair of quartic homogeneous equations for X (we set $\lambda = 1$ for conciseness in the following equations (3.11) and (3.12)):

$$XX_{\alpha\alpha\beta}(X_\alpha X_{\alpha\beta} + XX_\beta) + XX_{\alpha\alpha}(X^2 - X_{\alpha\beta}^2) = (X^2 + X_\alpha^2)X_{\alpha\beta}^2 + XX_\alpha X_\beta X_{\alpha\beta} - X^4 \tag{3.11}$$

plus another symmetrical equation obtained by interchanging the roles of α and β. Transforming to variable $x \equiv \ln X$, the result *is not* of a given parity, unlike the DB case. However, there is still the possibility that the even and odd parts *simultaneously vanish*, and can still be compatible with the system (3.10). This entails introducing the additional constraint:

$$x_{\alpha\beta}^2 = (x_\alpha^2 + 1)(x_\beta^2 + 1) \tag{3.12}$$

which can also be, returning to the original variable $X \equiv e^x$ and restoring the spectral parameter λ:

$$(X_\alpha^2/\lambda^2 - 2\cos\varphi X_\alpha X_\beta + \lambda^2 X_\beta^2) + X^2 \sin^2\varphi = 0. \tag{3.13}$$

The above constraint turns out to be indeed compatible with the system (3.10). One way to show it is to start from the two equations (3.13) and (3.10*c*) and note that the remaining equations (3.10*a*, *b*) immediately arise as consequences.

With the understanding that the pseudopotential is now restricted by (3.13) in addition to (3.10), the following Bäcklund transformation holds:

$$X' = 1/X \tag{3.14}$$

as in the DB case! The transformation formula for H is, since $H = -X_{\alpha\beta}/X$,

$$H + H' = -2X_\alpha X_\beta / X^2. \tag{3.15}$$

Again the BT appears to be reciprocal and the spectral parameter is hidden in the definition of the pseudopotential X.

The Painlevé expansion appears to be simpler in terms of H, which satisfies the equation:

$$H_{\alpha\beta} + HH_\alpha H_\beta/(1-H^2) = (1-H^2). \tag{3.16}$$

Its expansion is, retaining only the first few terms,

$$H = -\frac{2F_\alpha F_\beta}{F^2} + \frac{2F_{\alpha\beta}}{F} + u_0 + \ldots \tag{3.17}$$

where the functions F and u_0 are both arbitrary. The proposed BT may be rewritten in the form:

$$H' = -2X_\alpha X_\beta/X^2 + 2X_{\alpha\beta}/X + H \tag{3.18}$$

since $H \equiv -X_{\alpha\beta}/X$, and thus assumes the form of a truncated Painlevé expansion.

It must also be noted that the BT, in spite of its seemingly different aspect, actually coincides with the classical Bäcklund transformation.

3.3. An alternative formulation

We now turn to a variant of the preceding formulation of the sine-Gordon problem, which closely parallels the standard treatment of the Ernst equations and will be useful for the next section.

Let us perform a linear transformation on the two vectors $\boldsymbol{X}$ and $\boldsymbol{U}$, which become the new column vectors $\boldsymbol{A} = (A_1, A_2, A_3)$, $\boldsymbol{B} = (B_1, B_2, B_3)$:

$$\begin{aligned} &A_1 = X_1 + iX_2 \qquad &A_3 = X_1 - iX_2 \qquad &A_2 = iX_3 \\ &B_1 = U_1 - iU_2 \qquad &B_3 = U_1 + iU_2 \qquad &B_2 = -2iU_3 \end{aligned} \tag{3.19}$$

or, in matrix form,

$$(\boldsymbol{A}) = N(\boldsymbol{X})$$

$$(\boldsymbol{B}) = M(\boldsymbol{U}) \qquad M = (\det N)\tilde{N}^{-1}.$$

The system (3.2), or (3.3), becomes

$$\boldsymbol{B}_\alpha = -\boldsymbol{A} \wedge \boldsymbol{A}_\alpha \tag{3.20a}$$

$$\boldsymbol{B}_\beta = \boldsymbol{A} \wedge \boldsymbol{A}_\beta \tag{3.20b}$$

$$(A_1A_3 - A_2^2) = 1 \tag{3.20c}$$

still with the Euclidean definition of the cross-product.

The three equations: $\boldsymbol{X}_\alpha^2 = \lambda^2$, $\boldsymbol{X}_\beta^2 = 1/\lambda^2$, $\boldsymbol{X}_\alpha \cdot \boldsymbol{X}_\beta = H$ become, respectively,

$$[\tilde{\boldsymbol{A}}_\alpha]S[\boldsymbol{A}_\alpha] = \lambda^2$$

$$[\tilde{\boldsymbol{A}}_\beta]S[\boldsymbol{A}_\beta] = 1/\lambda^2$$

$$[\tilde{\boldsymbol{A}}_\beta]S[\boldsymbol{A}_\alpha] = H$$

where $S = \tilde{N}^{-1}N^{-1}$ and $[\tilde{A}]$ denotes a transposed vector. Thus all first-order α derivatives can be expressed in terms of any one of them, e.g. $A_{1\alpha}$; in particular, we obtain the following complete system of three equations for three unknowns A_1, B_3 and φ:

$$\begin{aligned} &A_{1\alpha}^2 + B_{3\alpha}^2 = -\lambda^2 A_1^2 \\ &A_{1\beta}^2 + B_{3\beta}^2 = -A_1^2/\lambda^2 \\ &A_{1\alpha}A_{1\beta} - B_{3\alpha}B_{3\beta} = -A_1^2 \cos\varphi. \end{aligned} \tag{3.21}$$

If we finally perform the non-linear transformation: $f = 1/A_1$, $\psi = A_2/A_1$, and note that ψ is the 'twist potential' associated with B_3, equations (3.21) take the form:

$$\begin{aligned} &(f_\alpha^2 + \psi_\alpha^2)/f^2 = -\lambda^2 \\ &(f_\beta^2 + \psi_\beta^2)/f^2 = -1/\lambda^2 \\ &(f_\alpha f_\beta + \psi_\alpha\psi_\beta)/f^2 = -\cos\varphi \end{aligned} \tag{3.22}$$

which is a limiting case of a formulation of the Ernst equations originally proposed by Cosgrove [21] (see § 4), where $\mathscr{E} \equiv f + \mathrm{i}\psi$ is the complex Ernst potential.

Now, the formulation (3.21), or (3.22), strongly suggests the introduction of *two* angular variables a, b—instead of the single angular variable φ considered up to here—defined as

$$\begin{aligned} &A_{1\alpha} = \mathrm{i}\lambda A_1 \cos a \qquad A_{1\beta} = -(\mathrm{i}/\lambda)A_1 \cos b \\ &B_{3\alpha} = \mathrm{i}\lambda A_1 \sin a \qquad B_{3\beta} = -(\mathrm{i}/\lambda)A_1 \sin b \end{aligned} \tag{3.23}$$

where λ may be complex. In terms of these, the sine-Gordon system may be very simply reformulated as the pair of equations:

$$\begin{aligned} &a_\beta = -(\mathrm{i}/\lambda)\sin b \\ &b_\alpha = \mathrm{i}\lambda \sin a. \end{aligned} \tag{3.24}$$

It is easily seen that both φ and $\tilde{\varphi}$:

$$\begin{aligned} &\varphi \equiv a + b \\ &\tilde{\varphi} \equiv a - b \end{aligned} \tag{3.25}$$

satisfy the sine-Gordon equation. The classical Bäcklund transformation is obviously the (reciprocal) transformation that exchanges the roles of φ and $\tilde{\varphi}$; it may also be viewed as the transformation that changes the sign of b, while leaving a invariant.

This illustrates the fact that *the property of being* BT *greatly depends on the choice of the unknown functions*: a transformation that merely exchanges two variables is not usually considered to be a Bäcklund transformation! The point is that $\tilde{\varphi}$ plays the role of a pseudopotential when φ is given (and vice versa).

4. The Ernst equation

4.1. The Ernst and the Cosgrove formulations

Under the simplifying assumptions of stationarity and axisymmetry, the Einstein equations in the vacuum give rise to the Ernst equation [7]. In such a case it has been shown by Papapetrou [20] that there is no loss of generality in adopting a metric of the form:

$$ds^2 = \frac{1}{f}[e^{\Pi/2}(dz^2 + d\rho^2) + \rho^2 \, d\varphi^2] - f[dt - \omega \, d\varphi]^2 \tag{4.1}$$

and the Einstein equations are (see, e.g., [21])

$$\Delta \ln f + (f^2/\rho^2)(\nabla\omega)^2 = 0 \tag{4.2}$$

$$\operatorname{div}[(f^2/\rho^2)\nabla\omega] = 0 \tag{4.3}$$

plus two equations involving the metric coefficient Π†:

$$\begin{aligned} \Pi_z &= 2\rho\left(\frac{f_\rho f_z}{f^2} - \frac{f^2\omega_\rho\omega_z}{\rho^2}\right) \\ \Pi_\rho &= \rho[(f_\rho^2 - f_z^2)/f^2 - (\omega_\rho^2 - \omega_z^2)(f^2/\rho^2)]. \end{aligned} \tag{4.4}$$

The operators ∇, $\Delta \equiv (\nabla)^2$ and div have their three-dimensional meaning and operate on axisymmetric functions.

It is convenient to perform a complex transformation on the cylindrical coordinates ρ, z and replace it by

$$\begin{aligned} \alpha &= z + i\rho \\ \beta &= z - i\rho \end{aligned} \tag{4.5}$$

which play the role of 'characteristic coordinates'. We use coordinates α, β instead of ρ, z throughout the rest of this paper.

Ernst [7] introduced the twist potential ψ whose existence is ensured by equation (4.3), which is in conservation form:

$$\begin{aligned} \psi_\alpha &= -\frac{2f^2}{(\alpha-\beta)}\omega_\alpha \\ \psi_\beta &= \frac{2f^2}{(\alpha-\beta)}\omega_\beta. \end{aligned} \tag{4.6}$$

† According to Ernst's Lagrangian and the Noether theorem, Π is the potential that represents momentum conservation parallel to the symmetry axis.

In terms of ψ the system of equations is

$$f_{\alpha\beta}=\frac{(f_\alpha-f_\beta)}{2(\alpha-\beta)}+\frac{1}{f}(f_\alpha f_\beta-\psi_\alpha\psi_\beta)$$
$$\psi_{\alpha\beta}=\frac{(\psi_\alpha-\psi_\beta)}{2(\alpha-\beta)}+\frac{1}{f}(f_\alpha\psi_\beta+f_\beta\psi_\alpha) \qquad (4.7)$$

together with

$$\Pi_\alpha=\frac{(\alpha-\beta)}{f^2}(f_\alpha^2+\psi_\alpha^2)$$
$$\Pi_\beta=-\frac{(\alpha-\beta)}{f^2}(f_\beta^2+\psi_\beta^2). \qquad (4.8)$$

Ernst further introduced the complex potential $\mathscr{E}\equiv(f+\mathrm{i}\psi)$ and showed that it satisfies the equation:

$$(\mathscr{E}+\mathscr{E}^*)\Delta\mathscr{E}=2(\nabla\mathscr{E})^2 \qquad (4.9)$$

which is, in α, β coordinates,

$$(\mathscr{E}+\mathscr{E}^*)\left(\mathscr{E}_{\alpha\beta}+\frac{(\mathscr{E}_\beta-\mathscr{E}_\alpha)}{2(\alpha-\beta)}\right)=2\mathscr{E}_\alpha\mathscr{E}_\beta. \qquad (4.10)$$

Cosgrove [21] pointed out the importance of equations (4.8) defining the metric coefficient Π and proposed an alternative formulation of the Ernst equation, consisting of that pair of equations†, completed by

$$\Pi_{\alpha\beta}=-(f_\alpha f_\beta+\psi_\alpha\psi_\beta)/f^2. \qquad (4.11)$$

The great usefulness of this formulation comes from the fact that it is equivalent to the statement that Π_α, Π_β and $\Pi_{\alpha\beta}$ are the coefficients of a pseudospherical metric.

The Ernst equation is invariant under a duality transformation (S) [22] characterised by the transformation formulae:

$$f'=\frac{(\alpha-\beta)}{2f} \qquad \psi'=\omega \qquad \omega'=\psi$$
$$\Pi'=\frac{(\alpha-\beta)}{f^2}\Pi \qquad (4.12)$$

while leaving α and β invariant.

It also possesses an SL(2) symmetry group, denoted (G), which expresses invariance of the Einstein equations with respect to linear transformations of the coordinates t, φ [8]. The image of (G) by the duality (S) is another SL(2) group, denoted by (H), which comprises scaling transformations and pure imaginary translations of $\mathscr{E}$ and of $1/\mathscr{E}$ as well; the latter being Ehlers' gravitational duality rotation [23]. A three-dimensional linear representation has been derived in terms of the Ernst potential as (see [8])

$$A_1=1/f \qquad A_2=\psi/f \qquad A_3=(f^2+\psi^2)/f \qquad (4.13)$$

† It is of interest to note that Π is *dimensionless* and is given by (4.8) in conservation form. In such cases, the consideration of the variable e^{Π} is useful and may be viewed as a generalisation of the Cole–Hopf transform [19].

meaning that the elements of the group (H) operate linearly on A_1, A_2, A_3; Π is an invariant of the group.

4.2. *The 'hydrodynamical' formulation*

We now go back to equation (2.15)—which, we recall, may be derived from one-dimensional gas dynamics—and substitute for Γ the expression

$$\begin{aligned} \Gamma &= -2\mathrm{i}\rho/\boldsymbol{X}^2 \\ &\equiv (\beta-\alpha)/\boldsymbol{X}^2. \end{aligned} \tag{4.14}$$

In the same way as in § 3.1 we define $\boldsymbol{X}'$ the unit vector parallel to $\boldsymbol{X}$: $\boldsymbol{X}' = \boldsymbol{X}/\sqrt{\boldsymbol{X}^2}$, and rewrite equations (2.15) in terms of $\boldsymbol{X}'$. Forsaking the primes (rewriting $\boldsymbol{X}$ for $\boldsymbol{X}'$ from now on), they become†

$$\begin{aligned} \boldsymbol{U}_\alpha &= -(\alpha-\beta)\boldsymbol{X}\wedge\boldsymbol{X}_\alpha \\ \boldsymbol{U}_\beta &= (\alpha-\beta)\boldsymbol{X}\wedge\boldsymbol{X}_\beta \\ \boldsymbol{X}^2 &= 1 \end{aligned} \tag{4.15}$$

which is a complete system of equations for the unknowns $\boldsymbol{U}$ and $\boldsymbol{X}$. Its dual form, obtained by solving for $\boldsymbol{X}_\alpha$, $\boldsymbol{X}_\beta$, is

$$\begin{aligned} \boldsymbol{X}_\alpha &= \frac{1}{(\alpha-\beta)}\boldsymbol{X}\wedge\boldsymbol{U}_\alpha \\ \boldsymbol{X}_\beta &= -\frac{1}{(\alpha-\beta)}\boldsymbol{X}\wedge\boldsymbol{U}_\beta. \end{aligned} \tag{4.16}$$

The geometrical arrangement of the vectors is the same as in the sine-Gordon case; in particular, the trihedrals $\{\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{U}_\alpha\}$ and $\{\boldsymbol{X}, \boldsymbol{X}_\beta, \boldsymbol{U}_\beta\}$ are both orthogonal. Introducing the two scalars:

$$\begin{aligned} H &\equiv \boldsymbol{X}_\alpha\cdot\boldsymbol{X}_\beta \\ K &\equiv (\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{X}_\beta) \equiv \Omega \end{aligned} \tag{4.17}$$

we have from the integrability condition of $\boldsymbol{U}$:

$$\boldsymbol{X}_{\alpha\beta} = \frac{(\boldsymbol{X}_\alpha - \boldsymbol{X}_\beta)}{2(\alpha-\beta)} - H\boldsymbol{X}. \tag{4.18}$$

Using this result, it is easily shown that the pair of equations

$$\begin{aligned} \Pi_\alpha &= -(\alpha-\beta)\boldsymbol{X}_\alpha^2 \\ \Pi_\beta &= (\alpha-\beta)\boldsymbol{X}_\beta^2 \end{aligned} \tag{4.19}$$

is integrable and defines a potential Π—later to be identified with the Papapetrou metric coefficient. We note the relations:

$$\begin{aligned} H &= \Pi_{\alpha\beta} \\ (H^2+K^2) &= (\boldsymbol{X}_\alpha^2)(\boldsymbol{X}_\beta^2) = -\Pi_\alpha\Pi_\beta/(\alpha-\beta)^2 \end{aligned} \tag{4.20}$$

which show that H and K can both be expressed in terms of Π.

† The present formulation (4.15) may also be viewed as a special case of Kinnersley's [8] equation (8.8).

Now, using the general analysis of § 2.2, we expand the second derivatives of $\boldsymbol{X}$ on the base $\{\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{X}_\beta\}$ and obtain a system of the form (2.3), with coefficients

$$
\begin{aligned}
&a_0 = \frac{1}{2(\alpha-\beta)} \qquad a_1 = -\frac{1}{2(\alpha-\beta)} \qquad D = -H \\
&b_0 = \partial_\alpha(\ln K) + \frac{1}{2(\alpha-\beta)} \qquad c_0 = \partial_\beta(\ln K) - \frac{1}{2(\alpha-\beta)} \\
&b_1 = \Pi_\alpha/(\alpha-\beta) \qquad c_1 = -\Pi_\beta/(\alpha-\beta) \\
&B = \frac{1}{c_1}\left[H\partial_\alpha \ln\left(\frac{K}{H}\right) - \frac{b_1}{2(\alpha-\beta)}\right] \qquad C = \frac{1}{b_1}\left[H\partial_\beta \ln\left(\frac{K}{H}\right) + \frac{c_1}{2(\alpha-\beta)}\right].
\end{aligned} \tag{4.21}
$$

The resulting equation satisfied by the scalar coefficients H, K, Π is

$$
\partial_{\alpha\beta}(\ln K) + BC + H + \frac{5}{4(\alpha-\beta)^2} = 0. \tag{4.22}
$$

Since H and K can be eliminated using (4.20), that is in effect *an equation for the single unknown function* Π. We show in § 4.4 that it does coincide with Cosgrove's equation for Π (denoted γ in [21]).

Equation (4.22) can be reformulated in a way which makes its relation with the sine-Gordon equation more transparent, in terms of $\varphi \equiv \tan^{-1}(K/H)$:

$$
\varphi_{\alpha\beta} + K + \frac{\sin(2\varphi)}{(\alpha-\beta)^2} + \frac{1}{2}\left[\partial_\alpha\left(\frac{K}{\Pi_\alpha}\right) + \partial_\beta\left(\frac{K}{\Pi_\beta}\right)\right] = 0.
$$

The first two terms give the sine-Gordon equation far away from the symmetry axis.

4.3. *The Bäcklund transformation*

With our choice of dependent variables, the Bäcklund transformation turns out to coincide with the duality transformation (S) given by (4.12), with f replaced by $1/X$, i.e.

$$
\begin{aligned}
&\Pi' = (\alpha-\beta)X^2\Pi \\
&X' = 1/[(\alpha-\beta)X].
\end{aligned} \tag{4.23}
$$

As in the sine-Gordon case, the validity of the above transformation depends on the fulfilment of a non-linear (quadratic) constraint† on the pseudopotential X:

$$
(b_1 X_\beta^2 + 2HX_\alpha X_\beta + c_1 X_\alpha^2) - K^2X^2 = 0 \tag{4.24}
$$

generalising (3.13), in addition to the linear system (2.3) and (4.21). Equation (4.24) may also be written in the form

$$
(X_\beta \boldsymbol{X}_\alpha - X_\alpha \boldsymbol{X}_\beta)^2 + X^2(\boldsymbol{X}_\alpha \wedge \boldsymbol{X}_\beta)^2 = 0 \tag{4.25}
$$

more readily interpretable in geometrical terms.

The proof of compatibility of (4.24) with the system (2.3) and (4.21) lies in the observation that (2.3*a*, *b*) arise as consequences of (4.24) and (2.3*c*).

† When the problem is reduced to (third-order) linear ordinary differential equations, that constraint takes the form of a second-order non-linear Appell equation [21].

4.4. Correspondence with the Ernst and the Cosgrove formulations

We observe that the vector $\boldsymbol{X}_\alpha$ satisfies two scalar conditions:

$$\boldsymbol{X}\cdot\boldsymbol{X}_\alpha=0 \qquad \boldsymbol{X}_\alpha^2=-\Pi_\alpha/(\alpha-\beta)$$

so that one of its components only is free, and similarly for $\boldsymbol{X}_\beta$. This suggests representing $\boldsymbol{X}_\alpha$ and $\boldsymbol{X}_\beta$ in terms of two numbers, a and b. It turns out that the parametrisation assumes a simple form if we first perform the complex linear transformation on $\boldsymbol{X}$ (equation (3.19)) introduced in § 3.3, which transforms it into a new vector $\boldsymbol{A}$. The derivatives of $\boldsymbol{A}$ may then be parametrised in the following way:

$$\begin{aligned} A_{1\alpha}&=\left(\frac{\Pi_\alpha}{\alpha-\beta}\right)^{1/2}A_1\cos a & A_{1\beta}&=\left(\frac{\Pi_\beta}{\beta-\alpha}\right)^{1/2}A_1\cos b\\ A_{2\alpha}&=\left(\frac{\Pi_\alpha}{\alpha-\beta}\right)^{1/2}(A_2\cos a+\sin a) & A_{2\beta}&=\left(\frac{\Pi_\beta}{\beta-\alpha}\right)^{1/2}(A_2\cos b-\sin b).\end{aligned}\tag{4.26}$$

The vector $\boldsymbol{U}$ too is linearly transformed by (3.19) into another vector $\boldsymbol{B}$, whose first derivatives are given by formulae similar to (4.26); in particular, three simple relations generalising (3.21) are found which, after performing the non-linear transformation $A_1\equiv 1/f$, $A_2\equiv\psi/f$, become

$$\begin{aligned}&(f_\alpha^2+\psi_\alpha^2)/f^2=\Pi_\alpha/(\alpha-\beta)\\ &(f_\beta^2+\psi_\beta^2)/f^2=-\Pi_\beta/(\alpha-\beta)\\ &(f_\alpha f_\beta+\psi_\alpha\psi_\beta)/f^2=-H=-\Pi_{\alpha\beta}\end{aligned}\tag{4.27}$$

which is Cosgrove's system. This completes the proof that the equation we started from (4.15) is equivalent to the Ernst equation.

4.5. Conformal invariance and the spectral parameter

The conformal invariance of equation (4.15) is spoilt by the presence of the factor $(\alpha-\beta)$ but can be restored by rewriting it in the form:

$$\begin{aligned}&\boldsymbol{U}_\alpha=-\Gamma\boldsymbol{X}\wedge\boldsymbol{X}_\alpha \qquad \boldsymbol{U}_\beta=\Gamma\boldsymbol{X}\wedge\boldsymbol{X}_\beta \qquad \boldsymbol{X}^2=1\\ &\Gamma_{\alpha\beta}=0.\end{aligned}\tag{4.28}$$

The formulae derived in the preceding sections can be easily rewritten, without need for further calculation, in conformally invariant form by merely inserting factors $\gamma_\alpha\equiv\partial_\alpha\ln\Gamma$, $\gamma_\beta\equiv\partial_\beta\ln\Gamma$ where needed. Thus equations (4.21) and (4.22) become, respectively,

$$\begin{aligned}&a_0=-\gamma_\beta/2 \qquad a_1=-\gamma_\alpha/2 \qquad D=-H\\ &b_0=\partial_\alpha(\ln K)+\gamma_\alpha/2 \qquad c_0=\partial_\beta(\ln K)+\gamma_\beta/2\\ &b_1=\gamma_\alpha\Pi_\alpha \qquad c_1=\gamma_\beta\Pi_\beta\\ &B=\frac{1}{c_1}\left[H\partial_\alpha\ln\left(\frac{K}{H}\right)+\tfrac12 b_1\gamma_\beta\right] \qquad C=\frac{1}{b_1}\left[H\partial_\beta\ln\left(\frac{K}{H}\right)+\tfrac12 c_1\gamma_\alpha\right]\end{aligned}\tag{4.29}$$

$$\partial_{\alpha\beta}(\ln K)+BC+H-\tfrac54\gamma_\alpha\gamma_\beta=0.\tag{4.30}$$

Together with the equation $\Gamma_{\alpha\beta}=0$, which is, in terms of $\gamma\equiv\ln\Gamma$,

$$\gamma_{\alpha\beta}=-\gamma_\alpha\gamma_\beta\tag{4.31}$$

equation (4.30) constitutes a complete system for the two unknowns Π and γ. The essential simplifying feature, which permits the introduction of a spectral parameter, is that the function γ occurs *through the product* $\gamma_\alpha\gamma_\beta$ *only* in (4.30) and this product does not change under the conformal transformations:

$$\begin{aligned} \alpha' &= 1/(\alpha+k) \\ \beta' &= 1/(\beta+k). \end{aligned} \tag{4.32}$$

We have indeed

$$\Gamma' \equiv (\alpha'-\beta') = -\frac{(\alpha-\beta)}{(\alpha+k)(\beta+k)}.$$

Hence

$$\gamma'_\alpha = \frac{1}{(\alpha-\beta)\nu^2} \qquad \gamma'_\beta = -\frac{\nu^2}{(\alpha-\beta)} \qquad \text{where} \qquad \nu \equiv \left(\frac{\alpha+k}{\beta+k}\right)^{1/2}. \tag{4.33}$$

The arbitrary constant k does not enter equation (4.30) for the invariant unknown function Π, but it does appear as the spectral parameter in the linear equations (2.3) and (4.29) defining the pseudopotential $\boldsymbol{X}$.

For more details we refer the reader to Kramer and Neugebauer's review [24].

5. A generalised sine-Gordon equation

Comparing equations (3.22) and (4.27), it becomes clear that the sine-Gordon equation is nothing other than the limiting form of the Ernst equation far away from the symmetry axis: it may be recovered from (4.27) by substituting

$$\begin{aligned} &\alpha = \alpha_0 + x \\ &\beta = \beta_0 + y \\ &\Pi = \tfrac{1}{2}(\alpha-\beta)^2 + F(x, y) \qquad F_{xy} = H+1 \end{aligned} \tag{5.1}$$

and letting $\alpha_0, \beta_0 \to \infty$ while keeping the new coordinates x, y finite; the result is the sine-Gordon system in coordinates x, y.

In the same way, the limiting form of the axisymmetric Einstein–Maxwell equations (considered in § 6), when α_0, β_0 become large, is a generalised sine-Gordon equation, which is called here the sine-Gordon–Maxwell equation. Its precise form may be derived from the Einstein–Maxwell system by merely replacing all factors ρ, i.e. $(\alpha-\beta)$, by a constant†.

Following Kinnersley ([8], equation (8.8)), the Einstein–Maxwell problem is formulated in § 6 as

$$\begin{aligned} B^i_\alpha &= (\alpha-\beta) f^{ijk} A_j A_{k\alpha} \\ B^i_\beta &= -(\alpha-\beta) f^{ijk} A_j A_{k\beta} \end{aligned} \tag{5.2}$$

† Since the Einstein–Maxwell equations have been identified as a particular tridimensional non-linear σ model [25], the sine-Gordon–Maxwell equations must be the same, restricted to two dimensions.

where A_i, B_i are eight-dimensional vectors and f^{ijk} is a totally antisymmetric tensor whose components are also given in [8]. The sine-Gordon–Maxwell limit is thus

$$\begin{aligned} B^i_\alpha &= f^{ijk} A_j A_{k\alpha} \\ B^i_\beta &= -f^{ijk} A_j A_{k\beta}. \end{aligned} \tag{5.3}$$

We will often write for conciseness this system in the form

$$\begin{aligned} \boldsymbol{B}_\alpha &= \boldsymbol{A} \wedge \boldsymbol{A}_\alpha \\ \boldsymbol{B}_\beta &= -\boldsymbol{A} \wedge \boldsymbol{A}_\beta \end{aligned} \tag{5.4}$$

recalling that f^{ijk} is totally antisymmetric.

5.1. Basic geometrical and group properties

The system (5.2) is manifestly invariant under a symmetry group (H′), which has the SL(3) structure, and $\boldsymbol{A}$, $\boldsymbol{B}$ belong to an eight-dimensional representation of the group. Four of the components of $\boldsymbol{A}$ only are independent and they can be parametrised in the following way [8]:

$$\begin{aligned} &A_1 = 1/f \qquad A_2 = \psi/f \qquad A_3 = \mathscr{E}\mathscr{E}^*/f \qquad A_4 = \Phi/f \\ &A_5 = \Phi^*/f \qquad A_6 = \Phi\mathscr{E}^*/f \qquad A_7 = \Phi^*\mathscr{E}/f \qquad A_8 = 1 - 3\Phi\Phi^*/f \end{aligned} \tag{5.5}$$

in terms of two real parameters f, ψ and one complex, Φ; $\mathscr{E} \equiv (f - \Phi\Phi^* + \mathrm{i}\psi)$ is to be identified with the complex Ernst potential [26]. These are the eight covariant components. A metric g_{ij} can also be defined, whose components in the same base are [8]

$$g_{13} = 2 \qquad g_{47} = g_{56} = -g_{22} = 1 \qquad g_{88} = 3. \tag{5.6}$$

The tensor f^{ijk} appearing in (5.2) is totally antisymmetric and its components are given in [8] as

$$f^{123} = -f^{247} = -f^{256} = \tfrac{1}{2} \qquad f^{167} = f^{345} = f^{478} = -f^{568} = \mathrm{i}/2. \tag{5.7}$$

5.2. More concise formulation, based on Cosgrove's method

The formulation (5.2) is extremely concise, as long as one does not write down explicitly all eight components. It is possible to derive a formulation involving only four complex variables U_1, U_2, V_1, V_2 in the following way.

Cosgrove's [21] formulation starts with the three equations defining the derivatives Π_α, Π_β and $\Pi_{\alpha\beta}$ of the metric coefficient Π. In the Einstein–Maxwell case, these equations assume the form:

$$\begin{aligned} -\frac{f_\alpha^2 + \Psi_\alpha^2}{f^2} + \frac{4}{f}(V_\alpha^2 + W_\alpha^2) &= \boldsymbol{A}_\alpha^2 = -\frac{\Pi_\alpha}{\alpha - \beta} \\ -\frac{f_\beta^2 + \Psi_\beta^2}{f^2} + \frac{4}{f}(V_\beta^2 + W_\beta^2) &= \boldsymbol{A}_\beta^2 = \frac{\Pi_\beta}{\alpha - \beta} \\ -\frac{f_\alpha f_\beta + \Psi_\alpha \Psi_\beta}{f^2} + \frac{4}{f}(V_\alpha V_\beta + W_\alpha W_\beta) &= \boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta = \Pi_{\alpha\beta} \end{aligned} \tag{5.8}$$

where V, W are the real and imaginary parts of the complex potential Φ and Ψ_α, Ψ_β is an abbreviation for

$$\begin{aligned}\Psi_\alpha &\equiv \psi_\alpha + 2(VW_\alpha - WV_\alpha)\\ \Psi_\beta &\equiv \psi_\beta + 2(VW_\beta - WV_\beta).\end{aligned} \tag{5.9}$$

(It should be noted that the above pair is not integrable and the symbol Ψ, without a subscript, has no meaning.)

As mentioned above, the sine-Gordon–Maxwell limit is obtained by replacing $\Pi_\alpha/(\alpha-\beta)$ and $\Pi_\beta/(\alpha-\beta)$ by constants, and $\Pi_{\alpha\beta}$ may be relabelled as H.

The first two equations (5.8), being a sum of four squares, suggest the following parametrisation in terms of four complex numbers U_1, U_2, V_1, V_2:

$$\begin{aligned}U_1 &= \frac{1}{f}(\Psi_\alpha - \mathrm{i}f_\alpha) \qquad & U_2 &= \frac{2}{\sqrt{f}}(V_\alpha + \mathrm{i}W_\alpha)\\ V_1 &= \frac{1}{f}(\Psi_\beta - \mathrm{i}f_\beta) \qquad & V_2 &= \frac{2}{\sqrt{f}}(V_\beta + \mathrm{i}W_\beta)\end{aligned} \tag{5.10}$$

or, using the complex Ernst potentials $\mathscr{E}$, Φ,

$$\begin{aligned}U_1 &= \frac{(\mathscr{E}_\alpha + 2\Phi^*\Phi_\alpha)}{\mathrm{i}f} \qquad & U_2 &= \frac{2\Phi_\alpha}{\sqrt{f}}\\ V_1 &= \frac{(\mathscr{E}_\beta + 2\Phi^*\Phi_\beta)}{\mathrm{i}f} \qquad & V_2 &= \frac{2\Phi_\beta}{\sqrt{f}}.\end{aligned} \tag{5.11}$$

The parametrisation is thus

$$\begin{aligned}\mathscr{E}_\alpha &= \mathrm{i}fU_1 - \Phi^*\sqrt{f}\,U_2 \qquad & \Phi_\alpha &= \tfrac{1}{2}\sqrt{f}\,U_2\\ \mathscr{E}_\beta &= \mathrm{i}fV_1 - \Phi^*\sqrt{f}\,V_2 \qquad & \Phi_\beta &= \tfrac{1}{2}\sqrt{f}\,V_2.\end{aligned} \tag{5.12}$$

The three equations (5.8) then are, in the SGM limit,

$$\begin{aligned}&U_2U_2^* - U_1U_1^* = 1\\ &V_2V_2^* - V_1V_1^* = 1\\ &P + P^* = 2H\end{aligned} \tag{5.13}$$

where, by definition of P,

$$P \equiv U_2V_2^* - U_1V_1^*. \tag{5.14}$$

Thus H is the real part of P; we shall denote its imaginary part by K:

$$P \equiv H + \mathrm{i}K. \tag{5.15}$$

A complete system of equations for the unknowns $U_{1,2}$, $V_{1,2}$ only, excluding the Ernst potentials, is obtained by expressing the conditions of integrability of $\mathscr{E}$ and Φ, given by (5.12), and of, e.g., B_1 and B_4. The resulting system is

$$\begin{aligned}U_{1\beta} &= \frac{1}{2\mathrm{i}}(P - U_1V_1) \qquad & U_{2\beta} &= \tfrac{1}{4}\mathrm{i}[2U_1V_2 + U_2(V_1 + V_1^*)]\\ V_{1\alpha} &= \frac{1}{2\mathrm{i}}(P^* - U_1V_1) \qquad & V_{2\alpha} &= \tfrac{1}{4}\mathrm{i}[2U_2V_1 + V_2(U_1 + U_1^*)]\end{aligned} \tag{5.16}$$

with P defined by (5.14).

5.3. Projection on the tangent plane to (Σ); *the third invariant L*

Following the general method outlined in § 2.1, we now calculate the second derivatives of $\boldsymbol{A}$ and $\boldsymbol{B}$, expanding them on a base in the eight-dimensional space. First we will consider the projections on the hyperplane tangent to the SL(3)-invariant surface (Σ) which is the locus of $\boldsymbol{A}$ as the Ernst potentials $\mathscr{E}$, Φ are arbitrarily varied. Of course, $\boldsymbol{A}_\alpha$ and $\boldsymbol{A}_\beta$ belong to the tangent plane and it can be seen that $\boldsymbol{B}_\alpha$ and $\boldsymbol{B}_\beta$ are tangent vectors too, so that, taken all together, these four vectors span the four-dimensional hyperplane tangent to (Σ).

Before computing the projections, some geometrical results concerning the first derivatives are needed. By the very definition (5.12), the derivatives $\boldsymbol{A}_\alpha$, $\boldsymbol{A}_\beta$ are seen to be calculable in terms of U_1, U_2, V_1, V_2 and the Ernst potentials; the derivatives $\boldsymbol{B}_\alpha$, $\boldsymbol{B}_\beta$, which are given by (5.4), are similarly calculable too. The following relations between derivatives are then easily seen to hold:

$$\begin{aligned} &\boldsymbol{A}_\alpha^2 = \boldsymbol{A}_\beta^2 = \boldsymbol{B}_\alpha^2 = \boldsymbol{B}_\beta^2 = 1 \\ &\boldsymbol{A}_\alpha \cdot \boldsymbol{B}_\alpha = 0 = \boldsymbol{A}_\beta \cdot \boldsymbol{B}_\beta \\ &\boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta = H = -\boldsymbol{B}_\alpha \cdot \boldsymbol{B}_\beta \\ &\boldsymbol{A}_\alpha \cdot \boldsymbol{B}_\beta = -K = \boldsymbol{A}_\beta \cdot \boldsymbol{B}_\alpha \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &\boldsymbol{B}_\alpha \wedge \boldsymbol{B}_\beta = -\boldsymbol{A}_\alpha \wedge \boldsymbol{A}_\beta \\ &\boldsymbol{A}_\beta \wedge \boldsymbol{B}_\alpha = -\boldsymbol{A}_\alpha \wedge \boldsymbol{B}_\beta . \end{aligned} \tag{5.18}$$

The last two relations indicate that four of the cross-products only are independent; they are

$$\boldsymbol{A}_\alpha \wedge \boldsymbol{A}_\beta \qquad \boldsymbol{A}_\alpha \wedge \boldsymbol{B}_\beta \qquad \boldsymbol{A}_\alpha \wedge \boldsymbol{B}_\alpha \qquad \boldsymbol{A}_\beta \wedge \boldsymbol{B}_\beta .$$

These four vectors span a four-dimensional subspace orthogonal to (Σ); together with the first derivatives $\boldsymbol{A}_\alpha$, $\boldsymbol{A}_\beta$, $\boldsymbol{B}_\alpha$, $\boldsymbol{B}_\beta$ they form a base of the eight-dimensional representation.

The second derivatives $\boldsymbol{A}_{\alpha\beta}$, $\boldsymbol{B}_{\alpha\beta}$ can be explicitly determined:

$$\begin{aligned} &\boldsymbol{A}_{\alpha\beta} = \boldsymbol{A}_\alpha \wedge \boldsymbol{B}_\beta \\ &\boldsymbol{B}_{\alpha\beta} = -\boldsymbol{A}_\alpha \wedge \boldsymbol{A}_\beta . \end{aligned} \tag{5.19}$$

Next, the second derivative $\boldsymbol{A}_{\alpha\alpha}$ may be formally expanded as

$$\boldsymbol{A}_{\alpha\alpha} = a_0 \boldsymbol{A}_\alpha + a_1 \boldsymbol{A}_\beta + b_0 \boldsymbol{B}_\alpha + b_1 \boldsymbol{B}_\beta + \cdot \tag{5.20}$$

where the dot symbolises the part orthogonal to (Σ). Using the properties (5.17) and (5.18) we have

$$\begin{aligned} &\boldsymbol{A}_\alpha \cdot \boldsymbol{A}_{\alpha\alpha} = 0 \\ &\boldsymbol{A}_\beta \cdot \boldsymbol{A}_{\alpha\alpha} = H_\alpha \\ &\boldsymbol{B}_\alpha \cdot \boldsymbol{A}_{\alpha\alpha} = B_8 \\ &\boldsymbol{B}_\beta \cdot \boldsymbol{A}_{\alpha\alpha} = -K_\alpha . \end{aligned} \tag{5.21}$$

The third projection, $\boldsymbol{B}_\alpha \cdot \boldsymbol{A}_{\alpha\alpha} \equiv B_8$, remains *a priori* undetermined, meaning that the quantity B_8 is a *new* SL(3)-invariant scalar, in addition to the already known scalars H and K (the index 8 is a reminder of the dimensionality of the representation). This yields four equations for the coefficients of the expansion but, introducing two complex numbers

$$l \equiv a_0 + \mathrm{i}b_0 \qquad m \equiv -a_1 + \mathrm{i}b_1 \tag{5.22}$$

these can be rewritten as a pair of complex equations†:

$$\begin{aligned} lP - m &= P_\alpha \\ l - mP^* &= \mathrm{i}B_8. \end{aligned} \tag{5.23}$$

The determinant of this system is

$$\delta \equiv 1 - PP^*. \tag{5.24}$$

In the same way the derivative $\boldsymbol{A}_{\beta\beta}$ may be obtained in the form

$$\boldsymbol{A}_{\beta\beta} = c_0\boldsymbol{A}_{\alpha\alpha} + c_1\boldsymbol{A}_\beta + d_0\boldsymbol{B}_\alpha + d_1\boldsymbol{B}_\beta + \cdot \tag{5.25}$$

where c_0, c_1, d_0, d_1 are real and

$$\begin{aligned} \tilde{l} &\equiv (c_1 - \mathrm{i}d_1) = (\mathrm{i}C_8 - PP^*_\beta)/\delta \\ \tilde{m} &\equiv -(c_0 + \mathrm{i}d_0) = -(P^*_\beta - \mathrm{i}P^*C_8)/\delta \end{aligned} \tag{5.26}$$

and C_8 is a fourth SL(3)-invariant scalar.

From $\boldsymbol{A}_{\alpha\alpha}$, $\boldsymbol{B}_{\alpha\alpha}$ may be at once deduced, using the formula

$$\boldsymbol{B}_{\alpha\alpha} = \boldsymbol{A} \wedge \boldsymbol{A}_{\alpha\alpha} \tag{5.27}$$

substituting the expression (5.20) and noting that $\boldsymbol{A} \wedge \boldsymbol{X} \equiv 0$ whenever $\boldsymbol{X}$ is orthogonal to (Σ). We thus find

$$\boldsymbol{B}_{\alpha\alpha} = a_0\boldsymbol{B}_\alpha - a_1\boldsymbol{B}_\beta - b_0\boldsymbol{A}_\alpha + b_1\boldsymbol{A}_\beta \tag{5.28}$$

without any residual component orthogonal to (Σ)! By symmetry

$$\boldsymbol{B}_{\beta\beta} = -c_0\boldsymbol{B}_\alpha + c_1\boldsymbol{B}_\beta + d_0\boldsymbol{A}_\alpha - d_1\boldsymbol{A}_\beta. \tag{5.29}$$

The above equations (5.28) and (5.29) are *linear* in both $\boldsymbol{A}$ and $\boldsymbol{B}$, which will turn out to act as (linear) pseudopotentials (see § 5.4).

We now turn to the determination of the complete set of PDE satisfied by the four basic unknowns, the scalar coefficients H, K, B_8, C_8; this can be achieved through an examination of the conditions for integrability of $\boldsymbol{B}$, namely by requiring $\partial_\beta(\boldsymbol{B}_{\alpha\alpha}) = \partial_\alpha(\boldsymbol{B}_{\alpha\beta})$. Differentiating (5.28) and (5.19), we respectively obtain

$$\begin{aligned} \partial_\beta(\boldsymbol{B}_{\alpha\alpha}) &= (a_{0\beta}\boldsymbol{B}_\alpha - a_{1\beta}\boldsymbol{B}_\beta - b_{0\beta}\boldsymbol{A}_\alpha + b_{1\beta}\boldsymbol{A}_\beta) + (b_1\boldsymbol{A}_{\beta\beta} - a_1\boldsymbol{B}_{\beta\beta}) + (a_0\boldsymbol{B}_{\alpha\beta} - b_0\boldsymbol{A}_{\alpha\beta}) \\ \partial_\alpha(\boldsymbol{B}_{\alpha\beta}) &= \boldsymbol{B}_\alpha \wedge \boldsymbol{B}_{\alpha\beta} + \boldsymbol{B}_{\alpha\alpha} \wedge \boldsymbol{B}_\beta. \end{aligned} \tag{5.30}$$

From the property that, whenever $\boldsymbol{X}$ and $\boldsymbol{Y}$ are both tangent to (Σ), their cross-product $\boldsymbol{X} \wedge \boldsymbol{Y}$ is orthogonal to (Σ) , we note that the last term, $\boldsymbol{B}_{\alpha\alpha} \wedge \boldsymbol{B}_\beta$, is orthogonal to (Σ); then, projecting (5.30) parallel to (Σ), this term can be eliminated and the required equations of motion can be derived, equating to zero the coefficients of $\boldsymbol{A}_\alpha$, $\boldsymbol{A}_\beta$, $\boldsymbol{B}_\alpha$, $\boldsymbol{B}_\beta$. They are

$$P_{\alpha\beta} = \tfrac{1}{4}(1 + PP^* - 2P^2) - \frac{1}{\delta}[(P^*P_\alpha P_\beta - PL_\alpha L_\beta) - \mathrm{i}(L_\alpha P_\beta + L_\beta P_\alpha)] \tag{5.31}$$

$$L_{\alpha\beta} = \tfrac{1}{2}\mathrm{i}(P - P^*) \equiv -K. \tag{5.32}$$

(The coefficients B_8, C_8 are related to L by $B_8 = L_\alpha$, $C_8 = -L_\beta$.)

† The complex equations for l and m could be directly obtained through a consideration of the complex vector $\boldsymbol{A} + \mathrm{i}\boldsymbol{B}$, instead of separately considering $\boldsymbol{A}$ and $\boldsymbol{B}$.

Equations (5.31) and (5.32) are the desired SL(3)-invariant system of equations for two scalars: one complex (P), the other real (L). The system may be somewhat simplified through the transformation

$$P = Q\,e^{iL} \tag{5.33}$$

which gives

$$\partial_{\alpha\beta}(\ln Q) + \frac{Q_\alpha Q_\beta}{Q^2\delta} = \frac{\delta}{4P}. \tag{5.34}$$

5.4. Eight-dimensional pseudopotentials

The expression (5.20) of the projection of the second derivative $\boldsymbol{A}_{\alpha\alpha}$ may be completed by adding the component orthogonal to (Σ):

$$\boldsymbol{A}_{\alpha\alpha} = -\boldsymbol{A} \wedge \boldsymbol{B}_{\alpha\alpha} - \boldsymbol{A}_\alpha \wedge \boldsymbol{B}_\alpha.$$

Using the geometrical result that, for all $\boldsymbol{X}$, $\boldsymbol{A} \wedge \boldsymbol{X}$ is tangent to (Σ), we see that the two terms on the RHS are the parallel and orthogonal projections, respectively; therefore, we have

$$\boldsymbol{A}_{\alpha\alpha} = a_0\boldsymbol{A}_\alpha + a_1\boldsymbol{A}_\beta + b_0\boldsymbol{B}_\alpha + b_1\boldsymbol{B}_\beta - \boldsymbol{A}_\alpha \wedge \boldsymbol{B}_\alpha. \tag{5.35}$$

As already noted, the formulation may be appreciably simplified through the consideration of the complex vector $\boldsymbol{C} \equiv \boldsymbol{B} + i\boldsymbol{A}$ and its conjugate, $\boldsymbol{C}^* \equiv \boldsymbol{B} - i\boldsymbol{A}$.

Following the general method and notation of § 2.1, we then choose the following complex base vectors:

$$\begin{array}{llll} \boldsymbol{E}_1 = -\boldsymbol{C}_\alpha \wedge \boldsymbol{C}_\beta & \boldsymbol{E}_2 = \boldsymbol{C}^*_\alpha \wedge \boldsymbol{C}^*_\beta & \boldsymbol{E}_3 = \boldsymbol{C}_\alpha \wedge \boldsymbol{C}^*_\alpha & \boldsymbol{E}_4 = -\boldsymbol{C}_\beta \wedge \boldsymbol{C}^*_\beta \\ \boldsymbol{E}_5 = \boldsymbol{C}_\alpha & \boldsymbol{E}_6 = \boldsymbol{C}^*_\alpha & \boldsymbol{E}_7 = -\boldsymbol{C}^*_\beta & \boldsymbol{E}_8 = -\boldsymbol{C}_\beta. \end{array} \tag{5.36}$$

The α derivatives of $\boldsymbol{E}_5$–$\boldsymbol{E}_8$ are already known, being given by (5.35), (5.19) and (5.28); the derivatives of the remaining $\boldsymbol{E}_1$–$\boldsymbol{E}_4$ may then be reduced to cross-products involving first derivatives only, which can be computed without difficulty. The result is an eight-component ordinary differential equation of the general form (2.1a):

$$\begin{pmatrix} \boldsymbol{E}_{1\alpha} \\ \vdots \\ \boldsymbol{E}_{8\alpha} \end{pmatrix} = M \begin{pmatrix} \boldsymbol{E}_1 \\ \vdots \\ \boldsymbol{E}_8 \end{pmatrix} \tag{5.37}$$

where the matrix coefficients are functions of the four scalars P, P^*, B_8 and C_8, in the following way:

$$M = \left[\begin{array}{cccc|cccc} l & 0 & 0 & -m & P & -\frac{1}{2}P & 0 & -\frac{1}{2} \\ 0 & l^* & 0 & -m^* & \frac{1}{2}P^* & -P^* & \frac{1}{2} & 0 \\ -m^* & -m & (l+l^*) & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}P & -\frac{1}{2}P^* \\ \hline 0 & 0 & -\frac{1}{2} & 0 & l & 0 & -m & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & l^* & 0 & -m^* \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{5.38}$$

l and m being given by (5.23).

By the general symmetry (here called ($\tilde{T}$)) which exchanges α and β, U_i and V_i ($i = 1, 2$), P and P^*, B_8 and C_8, l and $\tilde{l}$, m and $\tilde{m}$, another ODE of the form (2.1b)

also holds:

$$\begin{pmatrix} E_{1\beta} \\ \vdots \\ E_{8\beta} \end{pmatrix} = \tilde{M} \begin{pmatrix} E_1 \\ \vdots \\ E_8 \end{pmatrix} \tag{5.39}$$

where $\tilde{M}$ is a matrix whose coefficients correspond to the coefficients of M by the above transformation.

The equations of motion (5.31) and (5.32) may be recovered by forming the commutator $[M, \tilde{M}]$, as in (2.2).

The pair of equations (5.37) and (5.39) is an overdetermined linear system for the column vector $(E_1, \ldots, E_8)$; each element E_i thus plays the role of pseudopotential when the four basic unknowns P, P^*, B_8, C_8 (which are independently determined by the equations of motion (5.31) and (5.32)) are given. Furthermore, a pair of eighth-order linear ODE for each pseudopotential E_{i0} may be derived through the elimination of the seven remaining base vectors E_i ($i \neq i_0$); the general solution possesses eight linearly independent solutions, which are the eight components of E_{i0}.

It turns out, fortunately, that one does not have to solve pairs of eighth-order equations in order to obtain pseudopotentials. The eight-dimensional representation separates into a product of fundamental representations of dimension 3 and a much simpler set of pseudopotentials may be found, which are given by pairs of ODE of the third order only; they are derived in the following subsection.

5.5. *Separability into a pair of fundamental (3D) representations*

The four quantities U_i and V_i, which are given in terms of A by (5.10) and (5.5), are pseudopotentials too. They present the drawback of occurring non-linearly, but they are easier to handle in explicit computations, being fewer in number than the eight E_i considered in the preceding subsection.

We will have to calculate the α derivatives of $U_{1,2}$ and the β derivatives of $V_{1,2}$ which are not given by (5.16); they are, however, related to the second derivatives $A_{\alpha\alpha}$, $A_{\beta\beta}$, which have already been determined. We thus find

$$\begin{aligned} U_{1\alpha} &= \tfrac{1}{2}\mathrm{i}(U_1^2 - 1) + \frac{1}{\Delta^*}(U_2^* P_\alpha - \mathrm{i} V_2^* L_\alpha) \\ U_{2\alpha} &= \tfrac{1}{4}\mathrm{i}\, U_2(3U_1 + U_1^*) + \frac{1}{\Delta^*}(U_1^* P_\alpha - \mathrm{i} V_1^* L_\alpha) \end{aligned} \tag{5.40}$$

where, by definition,

$$\Delta \equiv U_1 V_2 - U_2 V_1 \qquad \text{hence} \qquad \Delta\Delta^* \equiv -\delta. \tag{5.41}$$

Exchanging the coordinates α, β we similarly have, for the β derivatives of V_i,

$$\begin{aligned} V_{1\beta} &= \tfrac{1}{2}\mathrm{i}(V_1^2 - 1) - \frac{1}{\Delta^*}(V_2^* P_\beta^* + \mathrm{i} U_2^* L_\beta) \\ V_{2\beta} &= \tfrac{1}{4}\mathrm{i}\, V_2(3V_1 + V_1^*) - \frac{1}{\Delta^*}(V_1^* P_\beta^* + \mathrm{i} U_1^* L_\beta). \end{aligned} \tag{5.42}$$

The quantities U_i, V_i ($i = 1, 2$) are thus pseudopotentials (over)determined by equations

(5.16), (5.40) and (5.42); an essential simplifying property is that *the equations determining U_1, V_1 separate from the rest of this system.* Equations (5.16) for U_1, V_1 are already in separated form; the separability of equation (5.40) for $U_{1\alpha}$ arises from the fact that the factors U_2^*/Δ^*, V_2^*/Δ^* occurring in this equation may be rewritten in terms of U_1, V_1 as

$$\begin{aligned} U_2^*/\Delta^* &\equiv (V_1 - P^* U_1)/\delta \\ V_2^*/\Delta^* &\equiv (PV_1 - U_1)/\delta. \end{aligned} \tag{5.43}$$

The result of this substitution is a system equivalent to a second-order non-linear ODE for U_1 or V_1:

$$\begin{aligned} U_{1\alpha} &= \tfrac{1}{2}\mathrm{i}(U_1^2 - 1) + \frac{1}{\delta}[(\mathrm{i}L_\alpha - P^* P_\alpha) U_1 + (P_\alpha - \mathrm{i}PL_\alpha) V_1] \\ V_{1\alpha} &= \tfrac{1}{2}\mathrm{i}(U_1 V_1 - P^*) \end{aligned} \tag{5.44}$$

and a symmetrical system which may be deduced from the above by exchanging the roles of α and β.

We now take advantage of the manifest Lorentz invariance of the scalar equations of motion (5.31) and (5.32) and introduce a spectral parameter λ (which may be complex) in the above equation (5.44) and its $(\tilde{T})$-symmetrical equation, which become

$$\begin{aligned} U_{1\alpha} &= \lambda(U_1^2 - 1) + \mathrm{i}U_1 L_\alpha + \frac{Q_\alpha}{\delta}(V_1\,\mathrm{e}^{\mathrm{i}L} - Q^* U_1) \\ V_{1\alpha} &= \lambda(U_1 V_1 - P^*) \end{aligned} \tag{5.45}$$

$$\begin{aligned} U_{1\beta} &= \frac{1}{\lambda}(P - U_1 V_1) \\ V_{1\beta} &= \frac{1}{\lambda}(1 - V_1^2) - \mathrm{i}V_1 L_\beta + \frac{Q_\beta^*}{\delta}(U_1\,\mathrm{e}^{-\mathrm{i}L} - QV_1). \end{aligned} \tag{5.46}$$

The variables α, β have here been rescaled by factors of two in order to get rid of reducible factors of $\frac{1}{2}$ in the above equations; then the scalar equations (5.31) and (5.32), which are the compatibility conditions of (5.45) and (5.46), become

$$\partial_{\alpha\beta}(\ln Q) + \frac{Q_\alpha Q_\beta}{Q^2\delta} = \frac{\delta}{P} \tag{5.47a}$$

$$L_{\alpha\beta} = 2\mathrm{i}(P - P^*) \equiv -4K \tag{5.47b}$$

$$P \equiv Q\,\mathrm{e}^{\mathrm{i}L}. \tag{5.47c}$$

We use the form (5.47) of the equations in what follows.

The system (5.45) and (5.46) is non-linear, but it can be linearised through the transformation:

$$\begin{aligned} X_\alpha/X &= -\lambda U_1 - \tfrac{1}{4}\mathrm{i}L_\alpha \\ X_\beta/X &= V_1/\lambda + \tfrac{1}{4}\mathrm{i}L_\beta \end{aligned} \tag{5.48}$$

which produces a linear system of the general form (2.3) for the (number) X, with coefficients

$$\begin{aligned}
&a_0 = \tfrac{1}{4}\mathrm{i}L_\beta \qquad a_1 = -\tfrac{1}{4}\mathrm{i}L_\alpha \qquad D = -(H + \tfrac{1}{16}L_\alpha L_\beta)\\
&b_0 = \tfrac{1}{2}\mathrm{i}L_\alpha - \frac{Q^* Q_\alpha}{\delta} \qquad c_0 = -\left(\tfrac{1}{2}\mathrm{i}L_\beta + \frac{QQ^*_\beta}{\delta}\right)\\
&b_1 = \lambda^2\left(1 + \frac{\mathrm{i}Q_\alpha\,\mathrm{e}^{\mathrm{i}L}}{4\delta}L_\beta\right) - \left(\frac{3L_\alpha^2}{16} + \frac{\mathrm{i}L_{\alpha\alpha}}{4} + \frac{\mathrm{i}Q^* Q_\alpha L_\alpha}{4\delta}\right)\\
&c_1 = \frac{1}{\lambda^2}\left(1 - \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}L}Q^*_\beta L_\alpha}{4\delta}\right) + \left(-\frac{3L_\beta^2}{16} + \frac{\mathrm{i}L_{\beta\beta}}{4} + \frac{\mathrm{i}QQ^*_\beta L_\beta}{4\delta}\right)\\
&B = -\lambda^2\,\mathrm{e}^{\mathrm{i}L}Q_\alpha/\delta \qquad C = -\frac{1}{\lambda^2}\mathrm{e}^{-\mathrm{i}L}\frac{Q^*_\beta}{\delta}.
\end{aligned} \tag{5.49}$$

The equations of motion (5.47) may be seen to arise by application of the general compatibility conditions (2.4) and (2.5).

A pair of third-order ODE for X can be deduced (see (2.7)), which present three linearly independent solutions X_1, X_2, X_3; these are the components of a three-dimensional vector $\boldsymbol{X}$, which provides an explicit *fundamental representation of the underlying SL(3) symmetry group, (H′)*.

Equations (5.47) are complex and their complex conjugate forms must be satisfied too; they may be viewed as the compatibility conditions of a set of equations for U_1^*, V_1^*, complex conjugate to the system (5.45) and (5.46) (the spectral parameter λ must be formally treated as pure imaginary under complex conjugation: $\lambda^* \equiv -\lambda$). The corresponding linearising transformation is the conjugate of (5.48) and the linear pseudopotential $\boldsymbol{X}^*$ (of which X^* is an arbitrary component) satisfies a system of the form (2.3) with coefficients which are the complex conjugates of those listed in (5.49). Thus the three-dimensional vector $\boldsymbol{X}^*$ provides another fundamental representation of (H′).

Still, the potentials X and X^* cannot be chosen completely independent from one another: there is a condition that they must satisfy, ensuring integrability of the remaining potentials U_2, V_2. This can be seen most simply from the fact that the ratio V_2/U_2 can be expressed in two ways against U_1, V_1 as

$$\frac{1 + V_1 V_1^*}{P + U_1 V_1^*} = \frac{V_2}{U_2} = \frac{P^* + V_1 U_1^*}{1 + U_1 U_1^*} \tag{5.50}$$

and hence the compatibility condition:

$$(PV_1 U_1^* + P^* U_1 V_1^*) - (U_1 U_1^* + V_1 V_1^*) = \delta. \tag{5.51}$$

The above constitutes a bilinear constraint on X and X^*; when, e.g., X is given, the constraint on X^* is linear and becomes

$$\frac{(PV_1 - U_1)}{\lambda}X^*_\alpha - \lambda(P^* U_1 - V_1)X^*_\beta - X^*\left(\delta + \frac{\mathrm{i}L_\alpha}{4\lambda}(PV_1 - U_1) + \frac{\mathrm{i}\lambda L_\beta}{4}(P^* U_1 - V_1)\right) = 0. \tag{5.52}$$

In the particular case where $L \equiv 0$ and P is real, the scalar equations reduce to the sine-Gordon equation (3.16), with $P \equiv H$; the potential X may then be chosen real, i.e. $X^* \equiv X$, and the bilinear constraint becomes the quadratic relation (3.13) which was found to be relevant to the sine-Gordon problem.

The Bäcklund transformation, based on linear pseudopotentials X, X^* satisfying the bilinear constraint, will be discussed in the following subsection.

5.6. *The Bäcklund transformation*

The sine-Gordon–Maxwell system studied here possesses an SL(2) group of symmetry, which corresponds to the group (G) of linear (t, φ) coordinate transformations [8] of the axisymmetric Einstein–Maxwell case (§ 6). Finite transformations of the group constitute Bäcklund transformations of our system (5.47).

Two of the three generators are trivial: the first, (G_1), represents infinitesimal translations of the metric coefficient ω (ω is related to a component of the eight-dimensional vector $\boldsymbol{B}$: $\omega \equiv -\frac{1}{2}B_1$), while leaving the Ernst potentials $\mathscr{E}$, Φ invariant. Another, (G_3), is the generator of scale transformations: $\delta\Phi = \Phi$, $\delta\mathscr{E} = 2\mathscr{E}$, $\delta\omega = -2\omega$. One generator only, (G_2), presents interesting transformation formulae:

$$\delta\Phi = \tfrac{1}{2}(B_4 - \Phi B_1) \qquad \delta\mathscr{E} = -\mathscr{E}B_1 - \Phi B_5 + \mathrm{i}B_2$$
$$\delta B_1 = \tfrac{1}{2}(B_1^2 - A_1^2) \qquad \delta A_1 = A_1 B_1 .$$

The resulting formulae for U_i, V_i are

$$\delta U_1 = \mathrm{i}U_1/f \qquad \delta U_2 = \mathrm{i}U_2/2f$$
$$\delta V_1 = -\mathrm{i}V_1/f \qquad \delta V_2 = -\mathrm{i}V_2/2f.$$

(G_1) and (G_3) obviously leave U_i and V_i invariant.

By exponentiation of (G_2), a finite transformation, corresponding to the interchange of t and φ coordinates in the Einstein–Maxwell case, may be derived:

$$f' = \omega^2 f + 1/4f$$
$$\omega' = -4\omega f^2/(1 + 4\omega^2 f^2). \tag{5.53}$$

In terms of the polar representation (σ, θ) of the complex number $(\omega + \mathrm{i}/2f)$, namely:

$$\sigma\,\mathrm{e}^{\mathrm{i}\theta} \equiv \omega + \mathrm{i}/2f \equiv \tfrac{1}{2}(\mathrm{i}A_1 - B_1) \tag{5.54}$$

the finite transformation also becomes

$$\sigma' = 1/\sigma \qquad \theta' = \pi - \theta \qquad f' = \sigma^2 f \qquad \omega' = -\omega/\sigma^2$$
$$\Phi' = \tfrac{1}{2}(B_1\Phi - B_4) \tag{5.55}$$

and

$$U_1' = U_1\,\mathrm{e}^{2\mathrm{i}\theta} \qquad U_2' = U_2\,\mathrm{e}^{\mathrm{i}\theta}$$
$$V_1' = V_1\,\mathrm{e}^{-2\mathrm{i}\theta} \qquad V_2' = V_2\,\mathrm{e}^{-\mathrm{i}\theta} \tag{5.56}$$

whence

$$P' = U_2 V_2^*\,\mathrm{e}^{2\mathrm{i}\theta} - U_1 V_1^*\,\mathrm{e}^{4\mathrm{i}\theta}$$
$$L' = L + 4\theta. \tag{5.57}$$

Equations (5.56) and (5.57), with θ determined by (5.54), define the proposed Bäcklund transformation. We show in the appendix that it generates the two-soliton formula, starting from the vacuum.

In order to be able to apply the BT we must now provide a means of calculating A_1, B_1 in terms of our basic unknowns P, L, U_i, V_i; the new potential θ will then be determined as

$$\tan\theta = -A_1/B_1. \tag{5.58}$$

Remembering the definition (5.48) of X, and equation (5.10) satisfied by f, we may identify

$$A_1 \equiv 1/f = -XX^*. \tag{5.59}$$

Rewriting X_1 for X, the two remaining components X_2, X_3 of $\boldsymbol{X}$ may also be identified with the Ernst potentials as

$$\Phi = X_2/X_1 \qquad \mathscr{E} = -X_3/X_1 \tag{5.60}$$

together with the complex conjugate relations; the resulting eight-dimensional representation is

$$\begin{aligned} &A_1 = -X_1X_1^* && A_2 = \tfrac{1}{2}\mathrm{i}(X_1X_3^* - X_3X_1^*) && A_3 = -X_3X_3^* \\ &A_4 = -X_2X_1^* && A_5 = -X_1X_2^* && A_6 = X_2X_3^* \\ &A_7 = X_3X_2^* && A_8 = 1 + 3X_2X_2^*. \end{aligned} \tag{5.61}$$

The two vectors $\boldsymbol{X}$ and $\boldsymbol{X}^*$ must satisfy a constraint: $(\mathscr{E} + \mathscr{E}^*) = 2(f - \Phi\Phi^*)$, i.e.

$$\tfrac{1}{2}(X_1X_3^* + X_3X_1^*) - X_2X_2^* = 1. \tag{5.62}$$

Using the three-dimensional restriction of the metric (5.6), this also becomes

$$\boldsymbol{X}\cdot\boldsymbol{X}^* = 1. \tag{5.63}$$

It is interesting to remark that, in the case where P is real ($P \equiv \cos\varphi$, $L \equiv 0$), equations (2.3) and (5.49) for the (real) vector potential $\boldsymbol{X}$ coincide with the sine-Gordon equations (3.6) and (3.19) for the vector potential $\boldsymbol{A}$ (denoted by $\boldsymbol{A}_{\mathrm{SG}}$ in what follows). Thus $\boldsymbol{X}$ may be identified with $\boldsymbol{A}_{\mathrm{SG}}$ and the constraint (5.63) reduces to $\boldsymbol{A}_{\mathrm{SG}}^2 = 1$, which are (3.3$c$) and (3.20$c$). According to (5.60), the Ernst potentials are then

$$\Phi \equiv \psi_{\mathrm{SG}} \qquad \mathscr{E} \equiv -(f_{\mathrm{SG}}^2 + \psi_{\mathrm{SG}}^2) \qquad \psi \equiv 0 \qquad f \equiv -f_{\mathrm{SG}}^2. \tag{5.64}$$

The sine-Gordon equation satisfied by $\varphi \equiv \cos^{-1} P$ is, using (5.47a) where $Q \equiv P$,

$$\varphi_{\alpha\beta} = -\sin\varphi. \tag{5.65}$$

On the other hand, the sine-Gordon–Maxwell system obviously reduces to sine-Gordon also in the case where the electromagnetic complex potential $\Phi \equiv 0$ and where, as a consequence, the eight-dimensional vector $\boldsymbol{A}$ possesses three non-vanishing components A_1, A_2 and A_3, which can be identified with the three-dimensional sine-Gordon vector potential $\boldsymbol{A}_{\mathrm{SG}}$. That case is characterised by the properties:

$$\delta \equiv 0 \qquad Q \equiv 1 \qquad P \equiv \mathrm{e}^{\mathrm{i}L} \qquad P^* \equiv 1/P. \tag{5.66}$$

We may identify $L \equiv \varphi$, $P \equiv \mathrm{e}^{\mathrm{i}\varphi}$, and using (5.47$b$) it can be seen without calculation that φ satisfies a sine-Gordon equation:

$$\varphi_{\alpha\beta} = -4\sin\varphi \tag{5.67}$$

with a normalisation *different by a factor* 4† from that in (5.65)!

We now go back to the Bäcklund transformation and the construction of a potential θ, given by (5.58). The vector $\boldsymbol{A}$ having been already determined in terms of $\boldsymbol{X}$, $\boldsymbol{X}^*$ in the form (5.61), the remaining quantity B_1 occurring in the definition (5.58) is obtainable by quadrature of

$$\begin{aligned} B_{1\alpha} &= 2\mathrm{i}(XX^*_\alpha - X^*X_\alpha) + XX^*L_\alpha \\ B_{1\beta} &= -2\mathrm{i}(XX^*_\beta - X^*X_\beta) + XX^*L_\beta. \end{aligned} \tag{5.68}$$

In the case where P is real and L is zero, X may be chosen real and B_1 identically vanishes. The formula (5.58) gives $\theta = \pi/2$ and the BT formula (5.57) reduces to $P' = -U_2V_2^* - U_1V_1^*$. In such a case the quantities U_i, V_i may be identified with the sine-Gordon angular variables a, b introduced in § 3.3, in the following way:

$$\begin{aligned} &U_1 \equiv \cos a \qquad && U_2 \equiv \mathrm{i} \sin a \\ &V_1 \equiv \cos b && V_2 \equiv -\mathrm{i} \sin b \\ &U_i^* = -U_i && V_i^* = -V_i \end{aligned} \tag{5.69}$$

and we have $P' \equiv \cos \varphi' = \cos \tilde{\varphi}$. Thus the proposed BT reduces, in this particular case, to the sine-Gordon classical Bäcklund transformation which exchanges φ and $\tilde{\varphi}$ (§ 3.3, equation (3.25)).

5.7. *Three-dimensional vector equations*

We have seen (§ 5.5) that the sine-Gordon–Maxwell system can be formulated as a pair of three-dimensional vector equations of the form (2.10) with coefficients satisfying (2.6) and (5.49). There are two vectors involved: one has been identified with the vector $\boldsymbol{X}$ considered in § 5.6, whose components determine the Ernst potentials (equation (5.60)); another, $\boldsymbol{U}$, is related to the complex conjugate $\boldsymbol{X}^*$ of $\boldsymbol{X}$, but not in a simple way. It thus appears desirable in the present case, in order to preserve the basic symmetry of complex conjugation, to rewrite the general system (2.10) in a modified form where the two unknown vectors are $\boldsymbol{X}$ and $\boldsymbol{X}^*$, rather than $\boldsymbol{X}$ and $\boldsymbol{U}$.

In view of the form of the constraint (5.62) and (5.63) it is best to replace $\boldsymbol{U}$ by a new vector (hereafter, denoted by $\boldsymbol{U}$ also) whose components are respectively

$$U^1 = X_3^*/2 \qquad U^2 = -X_2^* \qquad U^3 = X_1^*/2. \tag{5.70}$$

They may be viewed as the contravariant components of $\boldsymbol{X}^*$ (the upper index will also serve to distinguish the vector components U^i from the two variables U_i $(i = 1, 2)$ introduced in § 5.2).

The vector equations can be deduced starting from the identity

$$\Delta U_1^* \equiv (P^*U_2 - V_2) \tag{5.71}$$

where we substitute (using (5.11) and (5.60))

$$U_2 = \frac{\mathrm{i}}{\lambda}\sqrt{A_1}\,\partial_\alpha(X_2/X_1) \qquad V_2 = \frac{\lambda}{\mathrm{i}}\sqrt{A_1}\,\partial_\beta(X_2/X_1).$$

† Thus the transformation (5.64) which relates the sine-Gordon equation to the *real* sine-Gordon–Maxwell equation, unlike all other transformations considered in the present paper, also affects the value of the independent variables α, β, namely the transformation formulae (5.64) must be completed by $\alpha = 2\alpha_{\mathrm{SG}}$, $\beta = 2\beta_{\mathrm{SG}}$.

Also, using the definition (5.48) of X_1, we substitute

$$\lambda U_1^* = \frac{1}{U^3}(U_\alpha^3 + a_1 U^3).$$

Then the identity (5.71) is

$$\left(\frac{\Delta X_1^2}{\mathrm{i}\sqrt{A_1}\, U^3}\right)(U_\alpha^3 + a_1 U^3) = P^*(X_1 X_{2\alpha} - X_2 X_{1\alpha}) + \lambda^2 (X_1 X_{2\beta} - X_2 X_{1\beta}) \tag{5.72}$$

which is the third component of a vector equation:

$$\Omega(\boldsymbol{U}_\alpha + a_1 \boldsymbol{U}) = P^* \boldsymbol{X} \wedge \boldsymbol{X}_\alpha + \lambda^2 \boldsymbol{X} \wedge \boldsymbol{X}_\beta. \tag{5.73}$$

Another equation is obtained by exchanging the roles of α and β:

$$-\Omega(\boldsymbol{U}_\beta + a_0 \boldsymbol{U}) = \frac{1}{\lambda^2} \boldsymbol{X} \wedge \boldsymbol{X}_\alpha + P \boldsymbol{X} \wedge \boldsymbol{X}_\beta \tag{5.74}$$

and of course, we also have

$$\boldsymbol{U} \cdot \boldsymbol{X} = 1. \tag{5.75}$$

We proceed to analyse the main properties of the above system (5.73)–(5.75) without at first making any assumption on the coefficients. Since they occur in a homogeneous way, however, we can always set without loss of generality:

$$\Omega \equiv (\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{X}_\beta) \tag{5.76}$$

as in § 2.3. The system may be solved linearly for $\boldsymbol{X}_\alpha$, $\boldsymbol{X}_\beta$, in a form 'conjugate' to that of the original system:

$$\begin{aligned} -\Omega^*(\boldsymbol{X}_\alpha - a_1 \boldsymbol{X}) &= P \boldsymbol{U} \wedge \boldsymbol{U}_\alpha + \lambda^2 \boldsymbol{U} \wedge \boldsymbol{U}_\beta \\ \Omega^*(\boldsymbol{X}_\beta - a_0 \boldsymbol{X}) &= \frac{1}{\lambda^2} \boldsymbol{U} \wedge \boldsymbol{U}_\alpha + P^* \boldsymbol{U} \wedge \boldsymbol{U}_\beta \end{aligned} \tag{5.77}$$

where, by definition of Ω^* and δ,

$$-\Omega\Omega^* \equiv (1 - PP^*) = \delta. \tag{5.78}$$

It is also of interest to note the result:

$$-\mathrm{d}\boldsymbol{U} \cdot \mathrm{d}\boldsymbol{X} = (\lambda^2\, \mathrm{d}\alpha^2 - 2H\, \mathrm{d}\alpha\, \mathrm{d}\beta + \mathrm{d}\beta^2/\lambda^2) + (a_1\, \mathrm{d}\alpha + a_0\, \mathrm{d}\beta)^2 \tag{5.79}$$

where we have introduced: $H \equiv (P + P^*)/2$, $K \equiv (P - P^*)/2\mathrm{i}$.

We expand the second derivatives of $\boldsymbol{X}$ on the base $\{\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{X}_\beta\}$ in the usual way; we thus find

$$\boldsymbol{X}_{\alpha\alpha} = b_0 \boldsymbol{X}_\alpha + B \boldsymbol{X}_\beta + b_1 \boldsymbol{X}$$

where

$$b_1 = -(a_1 b_0 + a_0 B) + (a_{1\alpha} + a_1^2 + \lambda^2).$$

Similarly

$$\boldsymbol{X}_{\beta\beta} = C \boldsymbol{X}_\alpha + c_0 \boldsymbol{X}_\beta + c_1 \boldsymbol{X}$$

where

$$c_1 = -(a_0 c_0 + a_1 C) + (a_{0\beta} + a_0^2 + 1/\lambda^2)$$

and

$$X_{\alpha\beta} = (\omega_\beta - c_0) + (\omega_\alpha - b_0)X_\beta + DX$$

where $\omega \equiv \ln \Omega$ as usual, and

$$D = \tfrac{1}{2}(a_{0\alpha} + a_{1\beta}) - (a_0\omega_\alpha + a_1\omega_\beta) + (a_0 b_0 + a_1 c_0) + (a_0 a_1 - H).$$

Furthermore, a_0, a_1, b_0 and c_0 satisfy the equations

$$a_{1\beta} - a_{0\alpha} = 2\mathrm{i}K \tag{5.80}$$
$$b_0/\lambda^2 - 2Hc_0 = -P\omega_\beta - P^*_\beta + (P - 2P^*)a_0 - \lambda^2 C + (\omega_\alpha - a_1)/\lambda^2$$
$$-2Hb_0 + \lambda^2 c_0 = -P^*\omega_\alpha - P_\alpha + (P^* - 2P)a_1 - B/\lambda^2 + \lambda^2(\omega_\beta - a_0).$$

The sine-Gordon–Maxwell system is, as we show below, entirely specified by further requiring

$$\begin{aligned} \omega_\alpha &= a_1 + b_0 \\ \omega_\beta &= a_0 + c_0 \end{aligned} \tag{5.81}$$

(as in (2.6)) and also $(a_{0\alpha} + a_{1\beta}) = 0$, which may be rewritten as

$$\begin{aligned} a_0 &= \mathrm{i}L_\beta/4 \\ a_1 &= -\mathrm{i}L_\alpha/4. \end{aligned} \tag{5.82}$$

There are thus four unknown functions: Ω, P, P^* and L, and a free (constant) parameter, λ.

The equations of motion are provided by the general conditions (2.4) and (2.5); the first equations ((2.4*a*), (2.5*a*)) yield

$$\omega_{\alpha\beta} + BC + H = 0. \tag{5.83}$$

The second equations ((2.4*b*), (2.5*b*)) give, respectively,

$$\begin{aligned} &\partial_\beta(P^*B) + \lambda^2(BC + P^*) = 0 \\ &\partial_\alpha(PC) + \frac{1}{\lambda^2}(BC + P) = 0 \end{aligned} \tag{5.84}$$

both of which can be integrated as

$$P^*B/\lambda^2 = \omega_\alpha - \mathrm{i}L_\alpha/4 - F'(\alpha)$$
$$\lambda^2 PC = \omega_\beta + \mathrm{i}L_\beta/4 - G'(\beta).$$

The resulting expressions for B and C are

$$B = -\lambda^2\frac{P}{\delta}[Q_\alpha/Q - F'(\alpha)]$$

$$C = -\frac{P^*}{\lambda^2\delta}[Q^*_\beta/Q^* - G'(\beta)]$$

where we have introduced $Q \equiv P\,\mathrm{e}^{-\mathrm{i}L}$, $Q^* \equiv P^*\,\mathrm{e}^{\mathrm{i}L}$. Finally equation (5.83) yields a second-order equation for P:

$$P\partial_{\alpha\beta}(\ln P) = \frac{B}{\lambda^2}\partial_\beta \ln(PP^*) + \delta(1 + BC/P^*) - 4\mathrm{i}KP \tag{5.85}$$

which, in terms of $q \equiv \ln Q$, is

$$q_{\alpha\beta} + (q_\alpha - F')(q_\beta + G')/\delta = \delta/P. \tag{5.86}$$

The potential L may be redefined as $L + \mathrm{i}(G - F)$ without affecting P or P^*. Therefore the arbitrary functions F and G are reducible and can be set equal to zero without loss of generality. Equation (5.86) then reduces, as announced, to the sine-Gordon-Maxwell equation:

$$q_{\alpha\beta} + q_\alpha q_\beta/\delta = \delta/P \tag{5.87}$$

(the equation $L_{\alpha\beta} = -4K$ is also satisfied by virtue of (5.80)).

In order to construct the Bäcklund transformation we must, as shown in § 5.6, introduce an appropriate generalisation of the complex vector $\boldsymbol{C} \equiv \boldsymbol{A} + \mathrm{i}\boldsymbol{B}$. In the present formalism the quantity that corresponds to the eight-dimensional vector $\boldsymbol{A}$ is, obviously, the tensor:

$$A_i^j \equiv -X_i U^j \tag{5.88}$$

the correspondence being given by (5.61) and (5.70). The quantity corresponding to $\boldsymbol{C}$ is a tensor C_i^j, defined by

$$\begin{aligned} C_{i\alpha}^j &= -U^j(X_{i\alpha} - a_1 X_i) \equiv U^j[\boldsymbol{U} \wedge (\boldsymbol{X} \wedge \boldsymbol{X}_\alpha)]_i \\ C_{i\beta}^j &= -X_i(U_\beta^j + a_0 U^j) \equiv X_i[\boldsymbol{X} \wedge (\boldsymbol{U} \wedge \boldsymbol{U}_\beta)]^j \end{aligned} \tag{5.89}$$

(this is easily shown to be integrable by cross-differentiation). A second tensor C_i^{*j} may also be defined as

$$C_{i\alpha}^{*j} = -X_i(U_\alpha^j + a_1 U^j)$$

$$C_{i\beta}^{*j} = -U^j(X_{i\beta} - a_0 X_i)$$

which is related to A_i^j by

$$C_i^j + C_i^{*j} = A_i^j.$$

Rewriting X for X_i, U for U^j, C for C_i^j, the Bäcklund transformation is, in the light of the results of § 5.6,

$$\begin{aligned} C' &= 1/C \\ C^{*\prime} &= 1/C^* \end{aligned} \tag{5.90}$$

$$\begin{aligned} X' &= X/\sqrt{CC^*} \\ U' &= U/\sqrt{CC^*} \end{aligned} \tag{5.91}$$

$$L' = L + 2\mathrm{i}\ln(C/C^*)$$

$$P' = -\frac{C^*}{C}P - \frac{C^* C_\alpha C_\beta}{C^2(C + C^*)}$$

$$P^{*\prime} = -\frac{C}{C^*}P^* - \frac{C C_\alpha^* C_\beta^*}{C^{*2}(C + C^*)}.$$

The component X of $\boldsymbol{X}$ may be chosen arbitrarily, but then there exists a linear condition to be satisfied by the component of $\boldsymbol{U}$, as shown in § 5.5 (equation (5.52)).

6. The axisymmetric stationary Einstein–Maxwell equations

Papapetrou's metric is still compatible with a non-zero electromagnetic vector potential, with components $A_z = A_\rho = 0$, $A_\varphi \equiv A$, $A_t \equiv V$ in coordinates z, ρ, φ, t. Ernst [26] introduced the twist potential W associated with A and the complex electromagnetic potential $\Phi \equiv V + \mathrm{i}W$, in addition to the twist potential ψ already existing in the vacuum case (§ 5.1). Ernst's formulation for the Einstein–Maxwell equations is then

$$f\Delta\mathscr{E} = (\nabla\mathscr{E} + 2\Phi^*\nabla\Phi)\cdot\nabla\mathscr{E}$$

$$f\Delta\Phi = (\nabla\mathscr{E} + 2\Phi^*\nabla\Phi)\cdot\nabla\Phi$$

$$\mathscr{E} = (f - \Phi\Phi^*) + \mathrm{i}\psi$$

which generalises Ernst's equation (4.9).

The analysis of this system closely parallels that of the sine-Gordon–Maxwell system studied in the preceding section (§ 5). Cosgrove's equations (5.8) and Kinnersley's formulation (5.2) in terms of a pair of eight-dimensional vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ have already been given in § 5. The relation between $\boldsymbol{A}$ and the Ernst potentials (5.5) remains unchanged, as well as equations (5.9)–(5.12).

It is worth remarking here that, as in the sine-Gordon–Maxwell limiting case, there are two ways in which the Einstein–Maxwell system reduces to the vacuum Ernst equation: the first of course occurs when the electromagnetic potential Φ identically vanishes; the second when ψ and, e.g., W simultaneously vanish. The transformation formulae relating these two cases are still given by (5.64):

$$\mathrm{i}f^{\mathrm{E}} = \sqrt{f} \qquad \psi^{\mathrm{E}} = V \qquad \Pi^{\mathrm{E}} = \Pi/4 \qquad H^{\mathrm{E}} = P = P^* \qquad K^{\mathrm{E}} = \sqrt{\delta} \tag{6.1}$$

(where the upper index E refers to the Ernst case: $\Phi^{\mathrm{E}} \equiv 0$) and they too involve rescaling the unit length by a factor 2:

$$\alpha^{\mathrm{E}} = \alpha/2 \qquad \beta^{\mathrm{E}} = \beta/2.$$

6.1. The U_i, V_i formalism

Equations (5.13) must be replaced by

$$\begin{aligned} N &\equiv (U_2U_2^* - U_1U_1^*) = -\Pi_\alpha/(\alpha-\beta) \\ M &\equiv (V_2V_2^* - V_1V_1^*) = \Pi_\beta/(\alpha-\beta) \\ P &\equiv (U_2V_2^* - U_1V_1^*) \\ H &\equiv (P+P^*)/2 = \Pi_{\alpha\beta}. \end{aligned} \tag{6.2}$$

The new variables U_i, V_i $(i = 1, 2)$ satisfy a system generalising (5.16):

$$\begin{aligned} U_{1\beta} &= \frac{1}{2\mathrm{i}}(P - U_1V_1) + \frac{(U_1 - V_1)}{2(\alpha-\beta)} \\ V_{1\alpha} &= \frac{1}{2\mathrm{i}}(P^* - U_1V_1) + \frac{(U_1 - V_1)}{2(\alpha-\beta)} \end{aligned} \tag{6.3}$$

$$\begin{aligned} U_{2\beta} &= \tfrac{1}{4}\mathrm{i}[2U_1V_2 + U_2(V_1 + V_1^*)] + \frac{(U_2 - V_2)}{2(\alpha-\beta)} \\ V_{2\alpha} &= \tfrac{1}{4}\mathrm{i}[2U_2V_1 + V_2(U_1 + U_1^*)] + \frac{(U_2 - V_2)}{2(\alpha-\beta)}. \end{aligned} \tag{6.4}$$

We note the relation:

$$V_{1\alpha} - U_{1\beta} = -K. \tag{6.5}$$

6.2. The eight-dimensional representation

The scalar products given in (5.17) become, in the Einstein–Maxwell case,

$$\begin{aligned} &\boldsymbol{A}_\alpha^2 = N \qquad &&\boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta = H \qquad &&\boldsymbol{A}_\beta^2 = M \\ &\boldsymbol{B}_\alpha^2 = (\alpha-\beta)^2 N \qquad &&\boldsymbol{B}_\alpha \cdot \boldsymbol{B}_\beta = -(\alpha-\beta)^2 H \qquad &&\boldsymbol{B}_\beta^2 = (\alpha-\beta)^2 M \\ &\boldsymbol{A}_\alpha \cdot \boldsymbol{B}_\alpha = 0 = \boldsymbol{A}_\beta \cdot \boldsymbol{B}_\beta \\ &\boldsymbol{A}_\alpha \cdot \boldsymbol{B}_\beta = -(\alpha-\beta)K = \boldsymbol{A}_\beta \cdot \boldsymbol{B}_\alpha. \end{aligned} \tag{6.6}$$

We use these constraints and their derivatives and project $\boldsymbol{A}_{\alpha\alpha}$, $\boldsymbol{B}_{\alpha\alpha}$ on the four-dimensional subspace spanned by the four derivatives $\boldsymbol{A}_\alpha$, $\boldsymbol{A}_\beta$, $\boldsymbol{B}_\alpha$, $\boldsymbol{B}_\beta$, in the same way as in § 5.3:

$$\begin{aligned} \boldsymbol{A}_{\alpha\alpha} &= a_0\boldsymbol{A}_\alpha + a_1\boldsymbol{A}_\beta + b_0\boldsymbol{B}_\alpha + b_1\boldsymbol{B}_\beta + \cdot \\ \boldsymbol{B}_{\alpha\alpha} &= -b_0(\alpha-\omega)^2\boldsymbol{A}_\alpha + b_1(\alpha-\beta)^2\boldsymbol{A}_\beta + [a_0 + 1/(\alpha-\beta)]\boldsymbol{B}_\alpha - a_1\boldsymbol{B}_\beta. \end{aligned} \tag{6.7}$$

As in (5.22) we introduce complex numbers:

$$\begin{aligned} l &= a_0 + \mathrm{i}(\alpha-\beta)b_0 \\ m &= -a_1 + \mathrm{i}(\alpha-\beta)b_1 \end{aligned} \tag{6.8}$$

and obtain a linear system for these unknowns l and m:

$$\begin{aligned} -Nl + P^*m &= -\tfrac{1}{2}N_\alpha - \mathrm{i}B_8/(\alpha-\beta) \\ Pl - Mm &= P_\alpha + (P-N)/2(\alpha-\beta) \end{aligned} \tag{6.9}$$

where

$$B_8 \equiv \boldsymbol{B}_\alpha \cdot \boldsymbol{A}_{\alpha\alpha} = (\alpha-\beta)(\boldsymbol{A}, \boldsymbol{A}_\alpha, \boldsymbol{A}_{\alpha\alpha}). \tag{6.10}$$

The determinant is

$$\begin{aligned} \delta &= (MN - PP^*) \\ &\equiv -\Delta\Delta^* \end{aligned} \tag{6.11}$$

(see (5.41) for the definition of Δ).

$\boldsymbol{B}_{\alpha\alpha}$, given by (6.7), is thus determined; the mixed derivative $\boldsymbol{B}_{\alpha\beta}$ is also independently known:

$$\boldsymbol{B}_{\alpha\beta} = -(\alpha-\beta)\boldsymbol{A}_\alpha \wedge \boldsymbol{A}_\beta + (\boldsymbol{B}_\beta - \boldsymbol{B}_\alpha)/2(\alpha-\beta) \tag{6.12}$$

and the condition of integrability $\partial_\beta(\boldsymbol{B}_{\alpha\alpha}) = \partial_\alpha(\boldsymbol{B}_{\alpha\beta})$, gives two PDE that must be satisfied by l and m:

$$\begin{aligned} l_\beta &= -m\tilde{m} - \frac{(H+3\mathrm{i}K)}{4} - \frac{3}{4(\alpha-\beta)^2} \\ m_\beta &= -m\tilde{l} + \frac{(m-l)}{2(\alpha-\beta)} - \frac{N}{4} - \frac{3}{4(\alpha-\beta)^2} \end{aligned} \tag{6.13}$$

where $\tilde{l}$ and $\tilde{m}$ are the quantities that correspond to l and m under the symmetry that exchanges α and β; they are given in terms of $C_8 = \boldsymbol{B}_\beta \cdot \boldsymbol{A}_{\beta\beta}$ by a linear system symmetrical to (6.9) and they satisfy two PDE symmetrical to (6.13):

$$\begin{aligned} \tilde{l}_\alpha &= -m\tilde{m} - \tfrac{1}{4}(H - 3\mathrm{i}K) - \frac{3}{4(\alpha-\beta)^2} \\ \tilde{m}_\alpha &= -l\tilde{m} + \frac{(\tilde{l} - \tilde{m})}{2(\alpha-\beta)} - \frac{M}{4} - \frac{3}{4(\alpha-\beta)^2}. \end{aligned} \tag{6.14}$$

We observe that l and $\tilde{l}$ satisfy the simple relation:

$$\tilde{l}_\alpha - l_\beta = 3\mathrm{i}K/2. \tag{6.15}$$

Equations (6.13) and (6.14), where l, m are given in terms of the four scalars Π, K, B_8 and C_8 by (6.9), and $\tilde{l}$, $\tilde{m}$ are given by a symmetrical equation, form a complete system and constitute a scalar formulation of the Einstein–Maxwell system we started from.

6.3. *Separability into a pair of fundamental representations*

The new scalar functions B_8, C_8 that we have introduced may serve to determine the α derivatives of U_i and the β derivatives of V_i, which are not given by (6.3) and (6.4); then U_i, V_i will play the role of pseudopotentials. As in § 5.5, *the equations giving* U_1, V_1 *separate* from those involving the remaining unknowns; we find

$$\begin{aligned} U_{1\alpha} &= \tfrac{1}{2}\mathrm{i}(U_1^2 - N) + (lU_1 - mV_1) \\ V_{1\alpha} &= \tfrac{1}{2}\mathrm{i}(U_1V_1 - P^*) + \frac{(U_1 - V_1)}{2(\alpha-\beta)} \end{aligned} \tag{6.16}$$

$$\begin{aligned} U_{1\beta} &= \tfrac{1}{2}\mathrm{i}(U_1V_1 - P) + \frac{(U_1 - V_1)}{2(\alpha-\beta)} \\ V_{1\beta} &= \tfrac{1}{2}\mathrm{i}(V_1^2 - M) + (\tilde{l}V_1 - \tilde{m}U_1). \end{aligned} \tag{6.17}$$

The system (6.16) and (6.17) can be linearised through the transformation:

$$\begin{aligned} X_\alpha/X &= U_1/2\mathrm{i} - \tfrac{1}{3}l \\ X_\beta/X &= V_1/2\mathrm{i} - \tfrac{1}{3}\tilde{l} \end{aligned} \tag{6.18}$$

which does define a new potential X, owing to (6.5) and (6.15).

The system (6.16) and (6.17) may then be viewed as defining the second derivatives $X_{\alpha\alpha}$, $X_{\alpha\beta}$, $X_{\beta\beta}$ of X in the general form (2.3); the coefficients assume the form, generalising (5.49):

$$\begin{aligned} &a_0 = \left(\frac{1}{2(\alpha-\beta)} - \frac{\tilde{l}}{3}\right) \qquad a_1 = -\left(\frac{1}{2(\alpha-\beta)} + \frac{l}{3}\right) \\ &D = -\left(\tfrac{1}{6}H + \tfrac{1}{9}l\tilde{l} - \tfrac{1}{3}m\tilde{m} + \frac{\tilde{l} - l}{6(\alpha-\beta)} - \frac{1}{4(\alpha-\beta)^2}\right) \\ &b_0 = l/3 \qquad c_0 = \tilde{l}/3 \\ &B = -m \qquad C = -\tilde{m} \\ &b_1 = \tfrac{2}{9}l^2 - \tfrac{1}{3}m\tilde{l} - \tfrac{1}{4}N - \tfrac{1}{3}l_\alpha \\ &c_1 = \tfrac{2}{9}\tilde{l}^2 - \tfrac{1}{3}\tilde{m}l - \tfrac{1}{4}M - \tfrac{1}{3}\tilde{l}_\beta. \end{aligned} \tag{6.19}$$

Another potential X^* may also be defined through the equations complex conjugate to (6.18).

Both potentials satisfy pairs of third-order ODE, as shown in § 2.2, and thus provide a pair of fundamental representations of the group (H′).

6.4. *The Bäcklund transformation*

The Bäcklund transformation is identified with the reciprocal transformation of the group (G) that exchanges t, φ coordinates; the analysis goes as in § 5.6.

Two generators, (G_1) and (G_3), leave all U_i and V_i invariant, whereas the remaining (G_2) has transformation formulae:

$$\begin{aligned} \delta U_1 &= \frac{\mathrm{i}(\alpha-\beta)U_1}{f} - \frac{1}{f} \\ \delta V_1 &= \frac{-\mathrm{i}(\alpha-\beta)V_1}{f} - \frac{1}{f} \\ \delta U_2 &= \mathrm{i}(\alpha-\beta)U_2/2f \\ \delta V_2 &= -\mathrm{i}(\alpha-\beta)V_2/2f. \end{aligned} \tag{6.20}$$

The requirement that $\Pi - 2\ln f$ be invariant under linear coordinate transformations entails

$$\delta\Pi = -2B_1. \tag{6.21}$$

By exponentiation of (G_2), the finite transformation that exchanges the roles of t and φ may be constructed. In terms of the complex number

$$\sigma\,\mathrm{e}^{\mathrm{i}\theta} \equiv \omega + \mathrm{i}(\alpha-\beta)/2f \tag{6.22}$$

its action on the physical variables is

$$\begin{aligned} \sigma' &= 1/\sigma \qquad \theta' = (\pi-\theta) \\ f' &= \sigma^2 f \qquad \omega' = -\omega/\sigma^2 \end{aligned} \tag{6.23}$$

$$\begin{aligned} \Pi' &= \Pi + 4\ln\sigma \\ U_1' &= \mathrm{e}^{2\mathrm{i}\theta}U_1 - \mathrm{e}^{\mathrm{i}\theta}/\sigma f \\ V_1' &= \mathrm{e}^{-2\mathrm{i}\theta}V_1 - \mathrm{e}^{-\mathrm{i}\theta}/\sigma f \\ U_2' &= \mathrm{e}^{\mathrm{i}\theta}U_2 \\ V_2' &= \mathrm{e}^{-\mathrm{i}\theta}V_2 \end{aligned} \tag{6.24}$$

and consequently

$$N' = N - \frac{4\sigma_\alpha}{(\alpha-\beta)\sigma} = N - \frac{1}{\sigma^2 f^2} + \frac{\mathrm{e}^{\mathrm{i}\theta}U_1 + \mathrm{e}^{-\mathrm{i}\theta}U_1^*}{\sigma f}$$

$$P' = \mathrm{e}^{2\mathrm{i}\theta}\left(U_2 V_2^* - \mathrm{e}^{2\mathrm{i}\theta}U_1 V_1^* + \frac{\mathrm{e}^{\mathrm{i}\theta}}{\sigma f}(U_1 + V_1^*) - \frac{1}{\sigma^2 f^2}\right).$$

It is interesting to remark that, in the particular case where P is identically real, X can be chosen real too, and consequently the pseudopotential B_1 is either zero or a constant. With $B_1 \equiv -2\omega$ null, the angular potential θ takes the simple value $\theta = \pi/2$ and the modulus σ is $\sigma = (\alpha-\beta)/2f$ (equation (6.22)); then the BT (6.23) reduces to $f' = (\alpha-\beta)^2/4f$. Applying the transformation (6.1), $f^{\mathrm{E}} \propto \sqrt{f}$, we obtain in the (vacuum) Ernst case:

$$f' \propto (\alpha-\beta)/f$$

which is the celebrated duality transformation (S) introduced in [22]. Thus the BT described here constitutes, in this sense, an electrovac generalisation of the duality transformation.

More accurately, the duality (S) itself may not be generalisable to the electromagnetic case, as pointed out by Kramer and Neugebauer [24], but its image by the transformation (6.1) does admit such a generalisation, which is our Bäcklund transformation.

6.5. An ($\boldsymbol{X}, \boldsymbol{X}^*$)-symmetrical formulation

According to the analysis of the preceding section, $X_1 \equiv X$ is an arbitrary component of a 3D vector $\boldsymbol{X}$ which satisfies a system of vector equations of the form (2.10), with coefficients expressed by (6.19). This system however is not manifestly symmetrical in X, X^*; we show here how this symmetry can be recovered.

From the properties (5.12) of the Ernst potentials, it follows that the two remaining components X_2, X_3 of $\boldsymbol{X}$ may be defined as in (5.60); we also introduce a dual vector $\boldsymbol{U}$ whose components U^j are the contravariant components of $\boldsymbol{X}^*$, as defined in (5.70), and obtain

$$\boldsymbol{U} \cdot \boldsymbol{X} = S = [(\alpha-\beta)\delta]^{-1/3} \tag{6.25}$$

$$(\boldsymbol{X}, \boldsymbol{X}_\alpha, \boldsymbol{X}_\beta) = \Omega = 1/[4(\alpha-\beta)^{1/2}]. \tag{6.26}$$

The 8D vector $\boldsymbol{A}$ is still given by the formulae (5.61), except for an extra normalisation factor S:

$$A_1 = -X_1 X_1^*/S \equiv -2U^3 X_1/S \tag{6.27}$$

etc. Using the identity (5.71), a pair of first-order vector equations for $\boldsymbol{U}$ and $\boldsymbol{X}$ can be derived, in the manner of § 5.7:

$$\begin{aligned} &4S^{-1}\Omega(\boldsymbol{U}_\alpha + b_0^* \boldsymbol{U}) = P^* \boldsymbol{X} \wedge \boldsymbol{X}_\alpha - N \boldsymbol{X} \wedge \boldsymbol{X}_\beta \\ &4S^{-1}\Omega(\boldsymbol{U}_\beta + c_0^* \boldsymbol{U}) = M \boldsymbol{X} \wedge \boldsymbol{X}_\alpha - P \boldsymbol{X} \wedge \boldsymbol{X}_\beta \\ &\boldsymbol{U} \cdot \boldsymbol{X} = S \end{aligned} \tag{6.28}$$

where $b_0^* \equiv l^*/3$, $c_0^* \equiv \tilde{l}^*/3$.

The above formulation (6.28) involves three scalar functions only: Π, b_0 and c_0. From Π, the coefficients N, M and H may be derived through their definitions (6.2); from b_0 and c_0, K may be derived as (see (6.15))

$$K = 2\mathrm{i}(b_{0\beta} - c_{0\alpha}). \tag{6.29}$$

Hence $P \equiv (H + \mathrm{i}K)$ and P^*; and δ and S (equation (6.11) and (6.25)). The Einstein–Maxwell system is then entirely specified by imposing three additional constraints on coefficients. The first two are that b_0, c_0 coincide with the corresponding coefficients of $\boldsymbol{X}_{\alpha\alpha}$, $\boldsymbol{X}_{\beta\beta}$ in their expansion (2.3); the third may be chosen to be the relation

$$D = BC/3 - a_0 a_1 - H/6 \tag{6.30}$$

between coefficients in (2.3).

Using the general integrability conditions (2.4) and (2.5), we then recover the expressions (6.19) for the scalar coefficients of the Einstein-Maxwell system.

The Bäcklund transformation studied in § 6.4 involves the components A_1, B_1 of $\boldsymbol{A}$ and $\boldsymbol{B}$, which in the present formulation become the tensors

$$A_i^j \equiv -S^{-1}X_iU^j \tag{6.31}$$

and B_i^j; they can be grouped together into *a single complex tensor* C_i^j:

$$C_i^j \equiv (\alpha-\beta)A_i^j + \mathrm{i}B_i^j \tag{6.32}$$

which is determined by quadrature of

$$\begin{aligned} C^j_{i\alpha} &= -S^{-1}(\alpha-\beta)U^j(X_{i\alpha} - a_1X_i) \\ C^j_{i\beta} &= -S^{-1}(\alpha-\beta)X_i(U^j_\beta - a_0^*U^j). \end{aligned} \tag{6.33}$$

Thus the BT formulae (6.22) (where $\omega \equiv -B_1/2$, $f \equiv 1/A_1$) and (6.23) reduce to the inversion of the potential C_i^j:

$$C_i'^j = 1/C_i^j \tag{6.34}$$

(for some particular choice of the indices i, j).

6.6. *The spectral parameter*

It is of course essential to have a spectral parameter in the Bäcklund transformation and in the definition of the pseudopotentials. As is well known (see, e.g., [24]), this parameter arises as a result of conformal invariance and of a particular symmetry of the scalar equations.

As in the vacuum case, conformal invariance may be restored by replacing factors $(\alpha-\beta)$ in Kinnersley's formulation (5.2) by an arbitrary solution of the two-dimensional Laplace equation:

$$\Gamma_{\alpha\beta} = 0. \tag{6.35}$$

The Γ dependence in all the equations given in § 6 can be established without calculation by merely inserting factors Γ_α, Γ_β wherever needed in order to preserve conformal invariance. Thus the correct generalisation of definitions (6.2) of N, M, H, δ is ($\gamma \equiv \ln \Gamma$)

$$N = -\gamma_\alpha\Pi_\alpha \qquad M = -\gamma_\beta\Pi_\beta \qquad H = \Pi_{\alpha\beta} \qquad \delta \equiv (MN - PP^*). \tag{6.36}$$

It is convenient to also introduce the following reduced quantities, which are conformal invariants:

$$\hat{N} \equiv \Pi_\alpha/\gamma_\alpha \qquad \hat{M} \equiv \Pi_\beta/\gamma_\beta \qquad \hat{P} \equiv -P/\gamma_\alpha\gamma_\beta \qquad \hat{\delta} \equiv (\hat{M}\hat{N} - \hat{P}\hat{P}^*) \equiv \frac{\delta}{(\gamma_\alpha\gamma_\beta)^2} \tag{6.37}$$

and the quantity

$$\delta' \equiv \gamma_\alpha\gamma_\beta\hat{\delta} = \delta/\gamma_\alpha\gamma_\beta \tag{6.38}$$

will be useful too. After some reduction, it turns out that the Einstein-Maxwell scalar equations can be conveniently formulated in terms of two complex conjugate functionals F, F^* of the two basic unknowns Π and K (and of Γ):

$$\begin{aligned} F &\equiv \hat{P}^2\partial_{\alpha\beta}(\ln P) + \mathrm{i}K\hat{P}^2 + \hat{P}(P+H) - (\Pi_\alpha\hat{P}_\beta + \Pi_\beta\hat{P}_\alpha)/2 + \delta'/4(\hat{P}-1) \\ F^* &\equiv \hat{P}^{*2}\partial_{\alpha\beta}(\ln P^*) - \mathrm{i}K\hat{P}^{*2} + \hat{P}^*(P^*+H) - (\Pi_\alpha\hat{P}^*_\beta + \Pi_\beta\hat{P}^*_\alpha)/2 + \delta'/4(\hat{P}^*-1). \end{aligned} \tag{6.39}$$

Two other quantities, G and $\tilde{G}$ are related algebraically (but not linearly) to the above functionals:

$$F=\frac{\tilde{G}}{P^*}\left[\frac{PG}{\delta'}-\delta'\partial_\alpha\left(\frac{\Pi_\alpha}{\delta'}\right)\right]$$
$$F^*=\frac{G}{P}\left[\frac{P^*\tilde{G}}{\delta'}-\delta'\partial_\beta\left(\frac{\Pi_\beta}{\delta'}\right)\right]. \tag{6.40}$$

The two scalar equations governing the two unknowns (Π, K) are then, in conformally invariant form

$$\frac{\hat{P}^{*2}}{\delta'}\partial_\beta\left(\frac{G}{\hat{P}^*}\right)=-\frac{\Pi_\alpha F^*}{\delta'}+\frac{\Pi_\alpha}{4}(\hat{P}^*-1)-\frac{\hat{P}^*_\alpha}{2}$$
$$\frac{\hat{P}^2}{\delta'}\partial_\alpha\left(\frac{\tilde{G}}{\hat{P}}\right)=-\frac{\Pi_\beta F}{\delta'}+\tfrac{1}{4}\Pi_\beta(\hat{P}-1)-\tfrac{1}{2}\hat{P}_\beta. \tag{6.41}$$

In the vacuum case (considered in § 4), G and $\tilde{G}$ vanish and the Ernst equations reduce to the comparatively simple form:

$$F=0 \qquad F^*=0 \text{ (vacuum)}. \tag{6.42}$$

The essential symmetry property underlying the presence of a spectral parameter is that the scalar system of equations (6.39)-(6.41) involves the harmonic function Γ *through the product* $\pi\equiv\gamma_\alpha\gamma_\beta$ *only* and this product remains the same for all Γ of the general form (see (4.33))

$$\Gamma=\text{constant}\times\frac{(\alpha-\beta)}{(\alpha+k)(\beta+k)} \tag{6.43}$$

so that a given solution $\Pi(\alpha,\beta)$, $K(\alpha,\beta)$ is compatible with any $\Gamma(\alpha,\beta)$ of the above family (6.43). The constant k, which does enter the equations defining the vector pseudopotentials, is the spectral parameter.

Let us mention finally that (multi)solitons may be constructed by repeated application of the BT in the way illustrated in the appendix, starting from the 'vacuum solution':

$$\Pi_0\equiv\tfrac{1}{2}(\alpha-\beta)^2 \tag{6.44}$$

(or, more generally, $\Pi_0=\Gamma^2/2$). This choice ensures that the solitons obtained will converge towards sine-Gordon-Maxwell solitons in the limit defined by (5.1).

7. Conclusion

The non-linear evolution equations which are completely integrable by IST always have a fundamental Lie group of symmetry (G) which, in combination with a discrete Bäcklund transformation, generates the infinite Lie group of symmetry and infinite number of conservation laws [1-5]; (G) is in practice a rather simple group, such as SL(2) or SL(3), or its complexified versions. We have tried to emphasise in the present work the importance of the role played by the scalar equations, which control the evolution of the scalar invariants of the Lie group (G). The following common features (among which several are of course already well known) of many completely integrable equations emerge from their consideration.

(i) The scalar equations *are (by definition) invariant under (G)* but they are the compatibility conditions for the existence of linear pseudopotentials T, which constitute a tensorial representation of (G). In all cases considered here, vector pseudopotentials have been found, with dimension $\leqslant 3$.

(ii) The scalar equations *are manifestly* Lorentz *invariant*, or invariant under a similar transformation group (Galilean, conformal transformations, etc) including a continuous parameter, λ, but the equations defining the pseudopotential T *are not*, so that an explicit dependence on λ, the spectral parameter, can be introduced in the definition of T.

(iii) The scalar equations *present a discrete reciprocal symmetry* (the BT), which consists of the inversion of a particular component of T; this component may (as in the Dodd-Bullough case) or may not (as in the sine-Gordon case) be completely arbitrarily chosen, depending on the particular symmetry of the problem. The reciprocal nature of the BT is merely formal, due to the fact that different components of T may be selected after each application of the BT—in other words, owing to the linear superposition principle which holds within the T representation. The BT produces the infinite series of multisoliton solutions when applied to the vacuum.

An interesting feature is that, although the scalar equations may be said to be manifestly invariant under (G), the group (G) remains in fact *a hidden symmetry* of these equations as long as the equations giving the pseudopotentials remain unnoticed; invariance arises from the fact that the latter are linear and uncoupled. The only invariance that is manifest in the scalar equations is that associated with the spectral parameter.

We have here restricted the use of the name of BT to those Bäcklund transformations which do alter the value of the scalar invariants, since they are our basic unknowns. In that case a transformation such as Kramer and Neugebauer's [24] transformation I_1 should not be called here BT (but this is of course only a matter of our present convention and vocabulary).

It does not appear customary to choose scalars as the unknowns, as we have done here; one of the main advantages of the present formulation lies in its unifying power (§ 2) and increased clarity†, with respect to other formulations where the basic unknowns are *not* taken to be scalars. In the latter case, Bäcklund transformations frequently have varied and complicated aspects, in contrast with our basic formula:

$$T' = 1/T \tag{7.1}$$

which proved to apply to all cases studied here.

It is worth noting that, as first shown in [6], the above formula (7.1) does provide a BT for the Dodd-Bullough equation, which had been thought not to possess a Bäcklund transformation. It is straightforward to show that the Korteweg-de Vries Bäcklund transformation is of the type (7.1) too.

The sine-Gordon-Maxwell system analysed in § 5, and its multisoliton solutions, are straightforward generalisations of the sine-Gordon equation and solitons; both are two-dimensional hyperbolic non-linear wave equations. This should help the understanding of what precisely are the (multi)solitons of the axisymmetric Einstein-Maxwell equations, of which sine-Gordon-Maxwell constitutes the limit far away from the symmetry axis.

† The tensorial nature of the pseudopotential is obscured in the usual formulations where the basic unknowns are *not* chosen to be scalars. As a result, the BT formulae tend to be much more complicated than our simple result, (7.1).

For the Einstein–Maxwell system, we have shown the *separability* of the standard eight-dimensional representation of the group (H′) into a pair of three-dimensional fundamental representations ($\boldsymbol{X}$ and $\boldsymbol{X}^*$), which have been explicitly constructed in §§ 6.3 and 6.5 (see also §§ 5.5 and 5.7), a result which could not be obtained in [8] and is, as far as I know, a new result.

We also comment (§ 6.4) on the electrovac generalisation of the duality transformation (S), introduced by Kramer and Neugebauer [22].

Appendix. The sine-Gordon–Maxwell interacting soliton pair

The purpose of this appendix is to show that the proposed BT (§ 5.6) does produce multisolitons when applied to the vacuum solution: $P=P^*=1$, $L=0$. We will only consider the first two iterations, i.e. we construct the interacting soliton pair.

A1. The sine-Gordon–Maxwell soliton

In order to apply the BT we first have to determine the potentials associated with the vacuum. As observed in § 5.6 (see the paragraph following (5.63)), the potential $\boldsymbol{X}$ coincides with the sine-Gordon potential $\boldsymbol{A}_{SG}$ whenever P is real. Thus an arbitrary component, X say, is determined by equations (3.10) and (3.22), where $f\equiv 1/X$. Note that the only non-vanishing equation (3.10) is (3.10c) and that $H\equiv\cos\varphi=1$. This system has two independent solutions only: $\exp[\pm(\lambda\alpha-\beta/\lambda)]$; we take advantage of the Lorentz invariance to choose here, without loss of generality, $\lambda=1$, and rewrite

$$\begin{aligned} X_1&\equiv 1/f_{SG}=(\sigma+\sigma_0)/\sqrt{\sigma}\\ X_2&\equiv(\psi/f)_{SG}=1/\sqrt{\sigma}\\ X_3&\equiv(f^2+\psi^2)/f_{SG}=1/\sqrt{\sigma}\qquad \sigma=\exp[2(\alpha-\beta)].\end{aligned}\tag{A1}$$

Furthermore we set $\sigma_0=1$ in what follows, as this amounts to a mere shift in phase or translation.

With the above potential X_1, application of the BT, with an arbitrary spectral parameter λ, produces the new solution:

$$\begin{aligned} e^{iL/2}&=\frac{(\sigma+1)^2+ik\sigma}{(\sigma+1)^2-ik\sigma}\\ Q&=\frac{(\sigma^2-6\sigma+1)+ik\sigma}{(\sigma+1)^2+ik\sigma}\\ P&=\frac{[(\sigma^2-6\sigma+1)+ik\sigma][(\sigma+1)^2+ik\sigma]}{[(\sigma+1)^2-ik\sigma]^2}\end{aligned}\tag{A2}$$

where k is an integration constant. When k is zero the sine-Gordon soliton is recovered; the above soliton may thus be viewed as its complexified version.

The potential $\boldsymbol{X}$ associated with the soliton (A2) is, in the real (sine-Gordon) case,

$$\begin{aligned} X_1&=\exp[(\lambda\alpha-\beta/\lambda)](\sigma+x_0^2)\qquad x_0\equiv(\lambda+1)/(\lambda-1)\\ X_2&=\exp[-(\lambda\alpha-\beta/\lambda)](\sigma+1/x_0^2)/(\sigma+1)\\ X_3&=(X_1X_2-1)^{1/2}=(x_0-1/x_0)\sqrt{\sigma}/(\sigma+1).\end{aligned}\tag{A3}$$

In the general case where k is non-zero, X is given by the general formulation (2.3) with coefficients given by (5.7); a pair of third-order linear ODE for X can be derived, which are of Fuchsian type in the independent variable σ; their general solution turns out to be calculable in closed form and is even polynomial when the singularity exponent has been factored out. The general solution can thus be found in a systematic way [27], through a consideration of the Riemann diagram for the singularity exponents. The three independent solutions of the α ODE (assuming β constant) are thus found as

$$X_1 = \frac{\sqrt{\sigma}}{(\sigma+1)(x^2-1)^{1/2}} \qquad X_2 = \frac{\sigma^{\lambda/2}[(\sigma+x_0^2)+x_0x(\sigma+1)]}{(\sigma+1)(x^2-1)^{1/2}} \qquad \text{where} \qquad x \equiv \frac{ik\sigma}{(\sigma+1)^2} \tag{A4}$$

and X_3 is obtained from X_2 by changing λ to $-\lambda$ (and hence x_0 to $1/x_0$). The three solutions that satisfy *both* the α and β ODE are obtained by merely replacing $\sigma^{\lambda/2}$ by $\exp[(\lambda\alpha-\beta/\lambda)]$; the solution X_1 remains unchanged and is a function of $(\alpha-\beta)$ only.

When $x=0$ (i.e. $k=0$) the sine-Gordon potentials (A3) are recovered.

A2. *The two-soliton formula*

Recalling that x as defined above is pure imaginary, a possible choice for the potentials X, X^* is

$$\begin{aligned} X &= \exp(\lambda\alpha-\beta/\lambda)[(\sigma+x_0^2)+x_0x(\sigma+1)]/(\sigma+1)(x^2-1)^{1/2} \\ X^* &= \exp(\lambda\alpha-\beta/\lambda)[(\sigma+x_0^2)-x_0x(\sigma+1)]/(\sigma+1)(x^2-1)^{1/2}. \end{aligned} \tag{A5}$$

In view of the complexity of the formulae, which already include a free parameter k (hidden in the variable x), we restrict ourselves in what follows to the case $\lambda=3$ (i.e. $x_0=2$), which may be considered typical.

The BT formula (5.68), with L as in (A2), determines the pseudopotential B_1 by quadrature; against the variable α, the integral becomes

$$B_1 = -12\mathrm{i}\exp(-2\beta/\lambda)\int \frac{[(\sigma^3+3\sigma^2-8)-2x^2(\sigma+1)(\sigma^2-2)]}{(\sigma+1)^3(x^2-1)^2}x\sigma^2\,\mathrm{d}\sigma. \tag{A6}$$

The solution satisfying *both* equations (5.68) is found to be

$$B_1 = 6\mathrm{i}\exp 2(\lambda\alpha-\beta/\lambda)\frac{x(\sigma^2-4)}{(\sigma+1)^2(x^2-1)}+\mathrm{i}l \tag{A7}$$

where l is another integration constant. However, it can be absorbed in the new variable:

$$\tau \equiv \frac{\exp 2(\lambda\alpha-\beta/\lambda)}{\mathrm{i}l} \qquad (\lambda=3) \tag{A8}$$

which plays the role of phase factor of the second soliton created by the BT. Thus the constant l merely represents a phase shift of the soliton. Finally the factor $\mathrm{e}^{2\mathrm{i}\theta}$ which is given by (5.54) and determines the transformation formulae (5.56) is

$$\mathrm{e}^{2\mathrm{i}\theta} = \left(\frac{x-1}{x+1}\right)\left(\frac{\tau[(\sigma+4)^2+4x(\sigma+1)^2]+(\sigma+1)^2(x+1)}{\tau[(\sigma+4)^2-4x(\sigma+1)^2]+(\sigma+1)^2(x-1)}\right). \tag{A9}$$

The formulae (5.57) then give the interacting two-soliton solution:

$$\begin{aligned} e^{iL'/2} = e^{2i\theta}\, e^{iL/2} &= \left(\frac{(\sigma+1)^2(x+1)+\tau[(\sigma+4)^2+4x(\sigma+1)^2]}{(\sigma+1)^2(1-x)-\tau[(\sigma+4)^2-4x(\sigma+1)^2]}\right) \\ Q' &= \frac{[(\sigma^2-6\sigma+1)+x(\sigma+1)^2]+\tau[(\sigma^2-24\sigma+16)+4x(\sigma+1)^2]}{(\sigma+1)^2(x+1)+\tau[(\sigma+4)^2+4x(\sigma+1)^2]}. \end{aligned} \tag{A10}$$

We observe that, as $\tau \to 0$, the original complex soliton (A2) is recovered and, as $\tau \to \infty$, the same soliton obtains but for a phase shift of amplitude $\ln 4$. When $\sigma \to (0, \infty)$ however, a 'unit modulus' sine-Gordon soliton of the type $P = e^{iL}$ is found, so that the solution (A10) does not represent the most general complex soliton pair†. This is due to the lack of generality of our choice (A5) for the pseudopotentials X, X^*, namely because we have the same phase factor in both. If that restriction is relaxed, that is to say, if we choose the most general linear combination of potentials (A4) compatible with the bilinear constraint (5.52), then the most general soliton pair can be derived.

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† It is interesting to note that, when $x = 0$ (i.e. $k = 0$), our formula (A10) represents the collision of two sine-Gordon solitons *with different* masses; the mass ratio is $\sqrt{4} = 2$ (see § 5.6, (5.65) and (5.66)).